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ILLINOIS UNIV AT URBANA-CHAMPAIGN DECISION AND CONTROL LAB F/G 12/1  
DESIGN OF OPTIMAL SYSTEMS WITH LOW SENSITIVITY TO SMALL TIME DE--ETC(U)  
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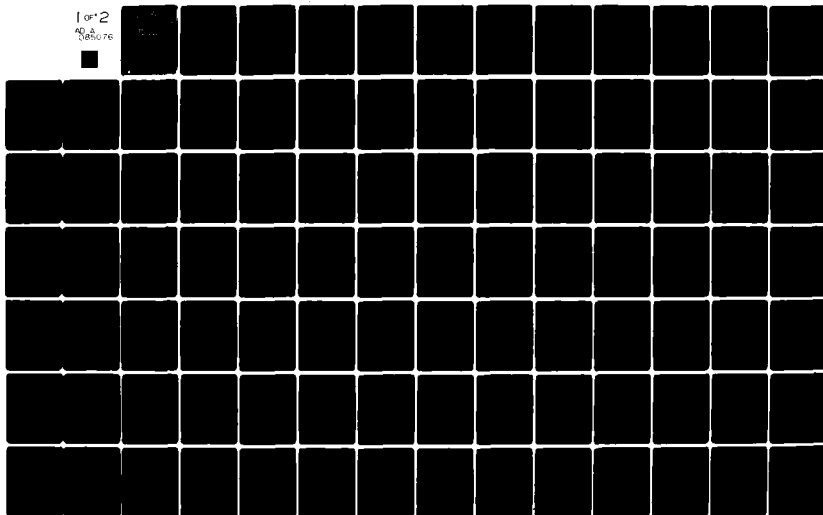
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JUNE, 1979

**CSL COORDINATED SCIENCE LABORATORY**

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**DESIGN OF OPTIMAL SYSTEMS  
WITH LOW SENSITIVITY  
TO SMALL TIME DELAYS**

MAHMOUD EL-SAYED SAWAN

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REPORT R-846

UILU-ENG 78-2239

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. AD A085076	3. RECIPIENT'S CATALOG NUMBER 9
4. TITLE (and Subtitle) 6 DESIGN OF OPTIMAL SYSTEMS WITH LOW SENSITIVITY TO SMALL TIME DELAYS.	5. TYPE OF REPORT & PERIOD COVERED Technical Report.	6. PERFORMING ORG. REPORT NUMBER R-846(DC-27); UILU-78-2239 ✓
7. AUTHOR(s) 10 MAHMOUD EL-SAYED/SAWAN	8. CONTRACT OR GRANT NUMBER(s) AFOSR-78-3632 N00014-79-C-0424 11 AFOSR-78-3632	10. PROGRAM ELEMENT PROJECT, TASK AREA & WORK UNIT NUMBERS
9. PERFORMING ORGANIZATION NAME AND ADDRESS Coordinated Science Laboratory University of Illinois at Urbana-Champaign Urbana, IL 61801	11. CONTROLLING OFFICE NAME AND ADDRESS Joint Services Electronics Program	12. REPORT DATE Jun-80 79
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (12) 112	15. SECURITY CLASS. (of this report) UNCLASSIFIED	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) (14) DC-27, UILU-ENG-78-2239		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Trajectory sensitivity, Small time delays, Nominal design, Optimal control systems, Deterministic systems, Stochastic systems, Design strategy, Augmented performance index, Minimum principle, Static minimization, Necessary conditions of optimality, Plant, Feedback path, State feedback control, Output feedback control, Reduced-order observer.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The work in this thesis is an addition to the study of trajectory sensitivity of optimal control systems. The parameter with respect to which sensitivity is studied is a small undesirable time delay that might occur in a system designed nominally with zero delays. A design strategy is proposed to guarantee low sensitivity of optimal control systems. A numerical algorithm is presented to solve the resulting necessary conditions of optimality. It is applied to a practical example. Encouraging results are obtained. ✦		

UILU-ENG 78-2239

DESIGN OF OPTIMAL SYSTEMS WITH LOW SENSITIVITY  
TO SMALL TIME DELAYS

by

Mahmoud El-Sayed Sawan

This work was supported in part by the U. S. Air Force under Grant AFOSR-78-3633 and in part by the Joint Services Electronics Program (U. S. Army, U. S. Navy, and U. S. Air Force) under Contract N00014-79-C-0424.

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DESIGN OF OPTIMAL SYSTEMS WITH LOW SENSITIVITY  
TO SMALL TIME DELAYS

BY

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B.S., University of Alexandria, 1973  
M.S., University of Alexandria, 1976

THESIS

Submitted in partial fulfillment of the requirements  
for the degree of Doctor of Philosophy in Electrical Engineering  
in the Graduate College of the  
University of Illinois at Urbana-Champaign, 1979

Thesis Advisor: J. B. Cruz, Jr.

Urbana, Illinois

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DESIGN OF OPTIMAL SYSTEMS WITH LOW SENSITIVITY  
TO SMALL TIME DELAYS

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Coordinated Science Laboratory  
Department of Electrical Engineering  
University of Illinois at Urbana-Champaign, 1979

The work in this thesis is an addition to the study of trajectory sensitivity of optimal control systems. The parameter with respect to which sensitivity is studied is a small undesirable time delay that might occur in a system designed nominally with zero delays. A design strategy is proposed to guarantee low sensitivity of optimal control systems. A numerical algorithm is presented to solve the resulting necessary conditions of optimality. It is applied to a practical example. Encouraging results are obtained.

## ACKNOWLEDGEMENT

The author wishes to express his sincere appreciation to his advisor, Professor J. B. Cruz, Jr. for his patient guidance and constant encouragement throughout the course of this thesis research. Sincere thanks are also due to Professors W. R. Perkins, P. V. Kokotovic, D. P. Looze, A. H. Haddad and J. Medanic for their helpful discussions. He also wishes to thank all the members of the Decision and Control Laboratory for their cooperation. Inexpressible thanks are given to his wife, Maha, for her innumerable tangible and intangible contributions which make her a full partner of this work. Special thanks are due to Mrs. Wilma McNeal and Mrs. Trudy Little for their expert typing of the manuscript.

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## CHAPTER 1

## INTRODUCTION

Small time-delays are usually expected to occur in systems due to several reasons, among which are the effect of mass and/or energy transport and the finite measuring time of the system outputs. These delays are very often neglected. However, they might cause significant deviation from the nominal system trajectory. Optimal systems with a delay have been studied by several authors. Chyung and Lee [2] obtained necessary and sufficient conditions of optimality and dealt with the questions of existence and uniqueness of the optimal control. Sannuti and Reddy [10] dealt with the problem by constructing an asymptotic power series solution in terms of time delay. That asymptotic approximation procedure was aimed at improving a 'nominal design' in which the small time delay is neglected. Mee [8] proposed a design method which calculates optimal linear feedback laws for the system, but ensures that the effect of control delays is kept small in closed loop. Inoue and Akashi [1] presented a synthesis method of a suboptimal control for a regulator problem, in which the optimal control is expanded into MacLaurin series in terms of the delay time and the first two or more terms are used to yield a suboptimal control. Kleinman [24] investigated systems with observation noise and showed that the optimal control for such systems can be generated by the cascade combination of a Kalman filter and a least mean squared predictor.

In this thesis we are concerned with the design of optimal control systems in a manner that makes their trajectories insensitive to small-time delays. The design strategy proposed here is to augment a standard quadratic performance index with a term of sensitivity measure. For deterministic systems, the minimization process is carried out using the well-

known Minimum Principle. This method was proposed by Stavroulakis and Sarachik [3], and extended to Discrete-time Systems by Cruz and Sawan [19]. Subbayyan and Varthilingam [20] reconsidered the same problem apparently without knowing about the work of Stavroulakis and Sarachik [3]. For stochastic systems, the minimization process of the augmented performance index is carried out by transforming the problem into an equivalent static minimization problem [22]. To apply the above methods to our problem of interest; sensitivity to small time-delays, we use the notation of a sensitivity function introduced by Inoue, Akashi, Ogino and Sawaragi [1].

Several cases are considered in this thesis. The delay is assumed to occur either in the plant (Chapters 3 and 5), or in the feedback path (Chapters 4 and 6). Several system structures are discussed. For each case, both stochastic and deterministic systems are studied. The proposed design strategy is applied to a practical example. The numerical results are analyzed and encouraging conclusions are obtained.

## CHAPTER 2

## PROBLEM FORMULATION AND METHODS OF ATTACK

2.1. Deterministic Systems

Systems considered in this thesis are of linear dynamics with feedback control. We consider the cases of state feedback and output feedback, as well as the case where state reconstruction is used. In the latter case we may use a full-order observer to reconstruct all the states, or a reduced-order observer to reconstruct the unmeasurable states only. The state equation is of the form

$$\begin{aligned}\dot{x}(t) &= A x(t) + C u(t) \\ \text{with } x(0) &= x_0\end{aligned}\tag{2.1}$$

where  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$  and  $A$  and  $C$  are matrices of appropriate dimensions. The output equation is

$$y(t) = E x(t)\tag{2.2}$$

where  $y(t) \in \mathbb{R}^l$  and  $E$  is a matrix of appropriate dimensions. The observer dynamic equation, if used is assumed to be of the form

$$\begin{aligned}\dot{z}(t) &= F z(t) + G y(t) \\ \text{with } z(0) &= z_0\end{aligned}\tag{2.3}$$

where  $z(t) \in \mathbb{R}^k$ ,  $k \leq n$  and  $F$  and  $G$  are matrices of appropriate dimensions.

We may have a state feedback control of the form

$$u(t) = D x(t),\tag{2.4}$$

an output feedback control of the form

$$u(t) = D y(t),\tag{2.5}$$

or, if an observer is used, a feedback control of the form

$$u(t) = D z(t) + K y(t) \quad (2.6)$$

where  $D$  and  $K$  are matrices of appropriate dimensions.

We will consider the different possibilities of occurrence of a delay in the system. If a delay,  $\mu$ , is expected to occur in the plant, the state equation will be of the form

$$\dot{x}(t) = A x(t) + B x(t-\mu) + C u(t) \quad (2.7)$$

$$\text{with } x(0) = x_0$$

along with the control law (2.4), (2.5), or (2.6). If a delay,  $\mu$ , is expected to occur in the feedback path, it will appear in the control law (2.4), (2.5), or (2.6) with the state equation (2.1) unchanged. If we have all the states available, we will use a state feedback control of the form

$$u(t) = D x(t-\mu) \quad (2.8)$$

If we have an output feedback control, we will use a control law of the form

$$u(t) = D E x(t-\mu) \quad (2.9)$$

If an observer is used, one of the following control laws will be applied:

$$u(t) = D z(t-\mu) + K y(t) \quad (2.10)$$

$$u(t) = D z(t) + K E x(t-\mu) \quad (2.11)$$

It should be noted that the control law (2.10) is not as likely to be expected as (2.11) since the observer is constructed on the control side. The purpose of considering that case is to cover all the possibilities of delay occurrence.

The problem under consideration is to design a system with zero time delay, which we will refer to as the nominal design, such that a quadratic performance index is minimized and sensitivity to small time delays is reduced. In other words, we require the trajectories of the system with the occurrence of a small time delay, as described above, to be as close as possible to the nominal trajectories, i.e. those of the system when the time delay is zero.

The standard linear regulator problem is to minimize a quadratic performance index of the form:

$$J = \frac{1}{2} \int_0^{\infty} (x'Qx + u'Ru)dt \quad (2.12)$$

where  $Q \geq 0$  and  $R > 0$  are matrices of appropriate dimensions, for  $\mu = 0$ . But if the actual system has some delay, due to some of the reasons discussed in the Introduction, and we use the standard problem solution, then the actual trajectories may be unacceptable compared to the nominal ones, obtained for zero delays. Reducing this sensitivity to small time delays may be achieved by trying to adjust  $Q$  and  $R$  in the standard linear regulator problem. For example decreasing  $R$  will result in a larger control which usually yields low sensitivity. However this approach is not convenient because it has no direct theoretical basis and it does not have a direct handle on the sensitivity of the different trajectories. In other words there is no systematic way to reduce the sensitivity of a particular trajectory of the system. All that can be done is to keep trying some different values of  $Q$  and  $R$  and observing the trajectories in each case to examine their sensitivity. In order to have a direct handle on sensitivity, we ought to define some sensitivity measure and incorporate it in the performance index to be minimized.

For our problem, sensitivity reduction can be achieved by defining the trajectory sensitivity function [1] as being the first partial derivative of the state vector with respect to the delay, evaluated at the nominal value of the delay, i.e. zero. Using the notation introduced above, we write

$$\sigma_x(t) = \left. \frac{\partial x(t, \mu)}{\partial \mu} \right|_{\mu=0} \quad (2.13)$$

where  $\sigma_x(t) \in \mathbb{R}^n$  and  $\sigma_x(0) = 0$ . To reduce the sensitivity with respect to the delay,  $\mu$ , the sensitivity function, defined by (2.13), is included in the performance index to be minimized with a suitable weighting matrix,  $S \geq 0$ , of appropriate dimensions. The augmented performance index to be minimized is of the form

$$J = E\left\{\frac{1}{2} \int_0^{\infty} (x'Qx + u'Ru + \sigma_x'S\sigma_x) dt\right\} \quad (2.14)$$

where  $E\{\cdot\}$  denotes the expected value over the initial condition  $x_0$ . This averaging process is required to avoid the dependence of the necessary conditions to be derived on the initial condition  $x_0$ .  $Q, R$  and  $S$  are design parameters to be adjusted to get a desirable behavior. It is noted that the choice of  $S$  with specific weighting is reflected directly on the sensitivity of the different state trajectories. This direct handle of  $S$  on sensitivity will be investigated in the examples presented in Chapter 7. It is noted that  $\sigma_z(t)$ , if an observer is used, is not introduced in the performance index,  $J$ , because what we practically care for is the plant trajectory  $x(t)$ .

Now we present a brief description of the minimization process which will be carried out to derive the necessary conditions for  $J$ , given by (2.14), to be minimal subject to a control law of one of the forms described above. The procedure consists of the following steps:

1. Obtain the dynamic equation of the closed loop sensitivity functions  $\sigma_x(t)$  and  $\sigma_z(t)$ , in case of using an observer, by calculating the first partial derivative of the state equation (2.1), (2.7) and (2.3) after applying the proper control law. Then substitute with the nominal value of the delay, i.e. zero, in all the equations.
2. Form the Hamiltonian, after applying the proper control law, as follows:

$$H = x'Qx + u'Ru + \sigma_x' S \sigma_x + \lambda_x \dot{x} + \lambda_{\sigma_x} \dot{\sigma}_x \quad (2.15)$$

or, if an observer is used,

$$H = x'Qx + u'Ru + \sigma_x' S \sigma_x + \lambda_x \dot{x} + \lambda_{\sigma_x} \dot{\sigma}_x + \lambda_z \dot{z} + \lambda_{\sigma_z} \dot{\sigma}_z \quad (2.16)$$

$\lambda_x$ ,  $\lambda_{\sigma_x}$ ,  $\lambda_z$  and  $\lambda_{\sigma_z}$  are appropriate Lagrange multipliers and are assumed to have the following relations:

$$\lambda_x = K_{11}x + K_{12}\sigma_x \quad (2.17a)$$

$$\lambda_{\sigma_x} = K_{21}x + K_{22}\sigma_x \quad (2.17b)$$

or, if an observer is used,

$$\lambda_x = K_{11}x + K_{12}z + K_{13}\sigma_x + K_{14}\sigma_z \quad (2.18a)$$

$$\lambda_z = K_{21}x + K_{22}z + K_{23}\sigma_x + K_{24}\sigma_z \quad (2.18b)$$

$$\lambda_{\sigma_x} = K_{31}x + K_{32}z + K_{33}\sigma_x + K_{34}\sigma_z \quad (2.18c)$$

$$\lambda_{\sigma_z} = K_{41}x + K_{42}z + K_{43}\sigma_x + K_{44}\sigma_z \quad (2.18d)$$

where  $K_{ij}$ ;  $i, j = 1, 2, 3, 4$  are constant matrices of appropriate dimensions.

3. Apply the minimum principle:

$$\dot{\lambda}_x = - \frac{\partial H}{\partial x} \quad (2.19a)$$

$$\dot{\lambda}_{\sigma_x} = - \frac{\partial H}{\partial \sigma_x} \quad (2.19b)$$

$$\dot{\lambda}_z = - \frac{\partial H}{\partial z} \quad (2.19c)$$

$$\dot{\lambda}_{\sigma_z} = - \frac{\partial H}{\partial \sigma_z} \quad (2.19d)$$

4. Equate  $\dot{\lambda}_x$ ,  $\dot{\lambda}_z$ ,  $\dot{\lambda}_{\sigma_x}$  and  $\dot{\lambda}_{\sigma_z}$  evaluated in (2.19a-d) to the corresponding values obtained by differentiating (2.18a-d) or (2.17a-b). Hence get equations for  $K_{ij}$ ;  $i, j = 1, 2, 3, 4$ . It will be seen that such equations are linear algebraic of the Lyapunov form.
5. The optimal feedback gain matrices satisfy the following necessary conditions:

$$0 = E \left\{ \int_0^{\infty} \frac{\partial H}{\partial D} dt \right\} \quad (2.20a)$$

$$0 = E \left\{ \int_0^{\infty} \frac{\partial H}{\partial K} dt \right\} \quad (2.20b)$$

where  $E\{\cdot\}$  denotes the expected value over the initial condition  $x_0$ .

It will be seen that, for given values of  $K_{ij}$ , the necessary conditions (2.20a-b) would generally result in linear equations of the form

$$D = \sum_{i=1}^N U_i D V_i + P \quad (2.21)$$

where  $U_i$ ,  $V_i$  and  $P$  are matrices of proper dimensions, evaluated in terms of  $K_{ij}$ ;  $i, j = 1, 2, 3, 4$ , and  $N$  is a nonnegative integer.

A numerical algorithm for solving equation (2.21) will be presented in section 2.3. It will also be seen that in the cases of the



delay occurrence in the plant, equation (2.21) will be as simple as

$$D = P \quad (2.22)$$

Since  $P$ ,  $U_i$  and  $V_i$  contain  $K_{ij}$ ;  $i, j = 1, 2, 3, 4$  which, in turn, depend on  $D$ , we use the following iterative algorithm to compute the optimal  $D$  and  $K$ , in case of using an observer:

1. Start with an initial guess for  $D$
2. Solve the Lyapunov-type equations to evaluate  $K_{ij}$ ;  $i, j = 1, 2, 3, 4$ .
3. Compute a new value of  $D$  from equation (2.22) or by solving equation (2.21).
4. Compare the new value of  $D$  with the old one. If it is close enough, stop. If it is not close enough, start a new iteration repeating steps 2 and 3 using the new value of  $D$ .

## 2.2. Stochastic Systems

Analogous to the deterministic systems considered in section 2.1, we will study linear dynamic systems with feedback controls, assuming disturbance occurrence in both the state equation and the output equation. The state equation in this case is

$$\dot{x}(t) = A x(t) + C u(t) + w_1(t) \quad (2.23)$$

$$\text{with } x(0) = x_0$$

where  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$ ,  $w_1(t) \in \mathbb{R}^n$  and  $A$  and  $C$  are matrices of appropriate dimensions. The output equation is

$$y(t) = E x(t) + w_2(t) \quad (2.24)$$

where  $y(t) \in \mathbb{R}^l$ ,  $w_2(t) \in \mathbb{R}^l$  and  $E$  is a matrix of appropriate dimensions.  $w_1(t)$  and  $w_2(t)$  are uncorrelated zero mean white noise processes with spectral density matrices  $\Sigma_1$  and  $\Sigma_2$  respectively. If an observer is used,

it is assumed to have the following dynamic equation

$$\dot{z}(t) = (A+B)z(t) + Cu(t) + F(y(t) - Ez(t)) \quad (2.25)$$

with

$$u(t) = Dz(t)$$

where  $z(t) \in \mathbb{R}^n$  and  $F$  and  $D$  are matrices of appropriate dimensions. We may use a state feedback control law of the form (2.4), an output feedback control law of the form

$$\begin{aligned} u(t) &= Dy(t) \\ &= DE x(t) \end{aligned} \quad (2.26)$$

Analogous to the study of deterministic systems, we will consider the different possibilities of occurrence of a delay in the system. If a delay,  $\mu$ , is expected to occur in the plant, the state equation will be of the form

$$\dot{x}(t) = A x(t) + B x(t-\mu) + C u(t) + w_1(t) \quad (2.27)$$

$$\text{with } x(0) = x_0$$

along with the control law (2.4), (2.25) or (2.26). If a delay,  $\mu$ , is expected to occur in the feedback path, the state equation (2.23) will remain unchanged along with one of the following control laws:

$$u(t) = Dx(t-\mu) \quad (2.28)$$

or

$$u(t) = DE x(t-\mu) \quad (2.29)$$

If an observer is used, we assume

$$y(t) = E x(t-\mu) + w_2(t) \quad (2.30)$$

along with equation (2.25). The case of a delay in the reconstructed state is also considered for the purpose of completeness. But it is not common

as discussed in Section 2.1. So, we will also consider a control law of the form

$$u(t) = Dz(t-\mu) \quad (2.31)$$

associated with the observer dynamics given by equation (2.25).

The standard linear regulator problem is to minimize a quadratic performance index of the form

$$J = \frac{1}{2} \lim_{T \rightarrow \infty} \frac{1}{T} E \left\{ \int_0^T (x'Qx + u'Ru) dt \right\} \quad (2.32)$$

where  $E\{\cdot\}$  denotes the expected value and  $Q$  and  $R$  are matrices of appropriate dimensions.

For our problem, low sensitivity can be achieved by incorporating the sensitivity function defined in equation (2.13) in a performance index. The augmented performance index to be minimized is of the form

$$J = \frac{1}{2} \lim_{T \rightarrow \infty} \frac{1}{T} E \left\{ \int_0^T (x'Qx + u'Ru + \sigma_x' S \sigma_x) dt \right\} \quad (2.33)$$

where  $S$  is a matrix of appropriate dimensions. As discussed in the above section, we do not have to introduce  $\sigma_z(t)$ , if an observer is used, in the performance index  $J$ .

Now, we present a brief description of the optimization procedure, which we will carry out to derive the necessary conditions of the performance index (2.33) to be minimized.

1. Derive the dynamic equation of the closed loop sensitivity functions  $\sigma_x(t)$  and  $\sigma_z(t)$ , in case of using an observer, by calculating the first partial derivative of the state equation (2.23) or (2.27) and (2.25), if an observer is used, after applying the proper control law. Then substitute with the nominal value of the delay, i.e., zero, in all

of the equations. Note that  $\frac{\partial w_1(t)}{\partial \mu} = \frac{\partial w_2(t)}{\partial \mu} = 0$ .

2. Form an augmented state vector

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ \sigma_x(t) \end{bmatrix} \quad (2.34)$$

or, if an observer is used,

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ z(t) \\ \sigma_x(t) \\ \sigma_z(t) \end{bmatrix} \quad (2.35)$$

Hence, form  $\bar{A}$ ,  $\bar{Q}$  and  $\bar{w}(t)$  such that the problem is reformulated as

$$\text{minimize } J = \frac{1}{2} \lim_{T \rightarrow \infty} \frac{1}{T} E \left\{ \int_0^T \bar{x}' \bar{Q} \bar{x} dt \right\} \quad (2.36)$$

with

$$\dot{\bar{x}} = \bar{A} \bar{x} + \bar{w} \quad (2.37)$$

and define

$$\bar{\Sigma} = E\{\bar{w} \bar{w}'\} \quad (2.38)$$

It is shown [22] that equations (2.36) and (2.37) can be expressed as an equivalent static minimization problem:

$$\text{minimize } J = \frac{1}{2} \text{tr}\{\bar{Q} P\} \quad (2.39)$$

such that  $P$  is the solution of the equation

$$\bar{A} P + P \bar{A}' + \bar{\Sigma} = 0 \quad (2.40)$$

It is also proved [22] that a necessary condition for  $J$ , given by (2.39)-(2.40), to be minimal is

$$0 = \frac{\partial J}{\partial D} = \frac{\partial}{\partial D} \text{tr}\{\bar{A} P K + \frac{1}{2} K \bar{\Sigma} + \frac{1}{2} \bar{Q} P\} \quad (2.41)$$

where

$$\bar{A}'K + K\bar{A} + \bar{Q} = 0 \quad (2.42)$$

$$\bar{A}P + P\bar{A}' + \bar{\Sigma} = 0 \quad (2.43)$$

If an observer is used, one more equation of the form (2.41) will be used to get the optimal value of  $F$ . Similar to what we got in the deterministic case, we have to use an iterative algorithm to compute the optimal feedback gains. That is because  $P$  and  $K$ , obtained by solving (2.42) and (2.43), depend on  $D$ . Then they are used in (2.41) to get  $D$ . The following algorithm will be used

1. Start with an initial guess for  $D$ .
2. Form  $\bar{A}$ ,  $\bar{Q}$  and  $\bar{\Sigma}$  and hence solve equations (2.42)-(2.43) to get  $P$  and  $K$ .
3. Solve equation (2.41) to obtain a new value of  $D$ .
4. Compare the new  $D$  with the old one. If it is close enough, stop.

If it is not close enough, start a new iteration repeating steps 2 and 3 using the new value of  $D$ .

### 2.3. Existence of a Solution

Let the necessary conditions be generally given by

$$f(D, S) = 0 \quad (2.44)$$

It is known [2] that for  $S = 0$ , there exists  $D^0$  such that  $f(D^0, 0) = 0$ . So, by the implicit function theorem, there exist open sets  $U \subset \mathbb{R}^{n \times m + n \times n}$  and  $W \subset \mathbb{R}^{n \times n}$  with  $(D^0, 0) \in U$  and  $0 \in W$  such that to every  $S \in W$  corresponds a unique  $D^*$  such that  $(D^*, S) \in U$  and  $f(D^*, S) = 0$ . In other words there exists  $D^*$  which satisfies (2.44) for any  $S$  in the neighborhood of 0.

The algorithms presented in Sections 2.1 and 2.2 are similar to the gain approximation algorithm which corresponds to the successive over relaxation (SOR) algorithm for the Gauss-Seidel iteration. It is proved [22] that the direction determined by that algorithm at the  $k$ th iteration is a

downhill direction. Thus this algorithm is in fact a descent method. The proofs for the cases considered in this thesis are similar to the proof presented in [22]. Step number 2 of both algorithms is to solve Lyapunov-type equations which can be done using the Riccati package of Linsys prepared by Bingulac [23]. Step number 3 of both algorithms is to solve an equation of the form (2.21). Here we propose an algorithm to solve that equation.

Consider the following iterative algorithm

$$D_{i+1} = \sum_{j=1}^N U_j D_i V_j + P \quad (2.45)$$

with  $D_0 = P$

To prove convergence of that algorithm we let  $\rho = \sup_j \{\rho(U_j), \rho(V_j)\}$ ;  $j = 1, \dots, N$  where  $\rho(U_j)$  and  $\rho(V_j)$  are the spectral radii of  $U_j$  and  $V_j$ , respectively. The error in the  $i^{\text{th}}$  iteration is defined as  $e_i = D_i - D_{i+1}$ , and its norm is found [7] to satisfy the following inequality

$$\|e_i\| \leq (i+1)^\gamma \cdot (N\rho^2)^{i+1} \cdot \Gamma \quad (2.46)$$

where  $\gamma$  and  $\Gamma$  are positive constants. If we assume that  $N\rho^2 < 1$ , the right hand side will go to zero as  $i \rightarrow \infty$ . So, the algorithm (2.45) converges under the sufficient condition

$$\rho < \frac{1}{\sqrt{N}} \quad (2.47)$$

#### 2.4. Advantages of this Design Scheme

The problem considered in this thesis is an extension of the standard Linear-Quadratic problem as described in sections 2.1 and 2.2.  $Q$ ,  $R$  and  $S$ , defined in equation (2.14), are design parameters to be adjusted in order to get a desirable behavior. The matrix  $Q$  is responsible for the

regulation of the states according to a desired weighting. The matrix  $R$  is responsible for minimizing the control effort. The matrix  $S$  is responsible for reducing trajectory sensitivity to a small time delay of the system states according to a desired weighting. This design scheme is reliable since it provides a guaranteed approach to reduce sensitivity in general. The choice of  $S$  with a specific weighting is reflected directly on the sensitivity of the different state trajectories. This direct handle on sensitivity is the main advantage of this proposed design scheme.

## CHAPTER 3

## DETERMINISTIC SYSTEMS WITH A TIME DELAY IN THE PLANT

3.1. State Feedback Control

Referring to the general problem formulation stated in Section 2.1, we have, in this case, a state equation (2.7) along with a state feedback control law (2.4) and a performance index (2.14).

The equations to be solved are summarized as follows:

$$K_{11}\tilde{A} + \tilde{A}'K_{11} = -\tilde{Q} + K_{12}\tilde{B}\tilde{A} + \tilde{A}'B'K_{21} \quad (3.1)$$

$$K_{12}\tilde{A} + \tilde{A}'K_{12} = \tilde{A}'B'K_{22} \quad (3.2)$$

$$K_{21}\tilde{A} + \tilde{A}'K_{21} = K_{22}\tilde{B}\tilde{A} \quad (3.3)$$

$$K_{22}\tilde{A} + \tilde{A}'K_{22} = -S \quad (3.4)$$

$$\tilde{A}M + M\tilde{A}' = -\Sigma \quad (3.5)$$

$$\tilde{A}G_1 + G_1\tilde{A}' = B\tilde{A}M \quad (3.6)$$

$$\tilde{A}G_2 + G_2\tilde{A}' = G_1\tilde{A}'B' + \tilde{B}\tilde{A}G_1' \quad (3.7)$$

$$D = R^{-1}C' \left\{ (B'K_{21} - K_{11})M + (B'K_{22} - K_{12})G_1 - K_{21}G_1' - K_{22}G_2 \right\} M^{-1} \quad (3.8)$$

where

$$\tilde{A} = A + B + CD \quad (3.9)$$

$$\tilde{Q} = Q + D'RD \quad (3.10)$$



and

$$\Sigma = E\{x_o x_o'\} \quad (3.11)$$

The above equations are obtained directly by applying the optimization procedure described in Section 2.1. Here we present the main steps of the derivation. For  $\mu = 0$ , we have [1]

$$\dot{x} = \tilde{A} x \quad (3.12)$$

$$\dot{\sigma} = \tilde{A} \sigma - B \tilde{A} x \quad (3.13)$$

and

$$J = \frac{1}{2} \int_0^{\infty} (x' \tilde{Q} x + \sigma' S \sigma) dt \quad (3.14)$$

where  $\tilde{A}$  and  $\tilde{Q}$  are defined above. The Hamiltonian is given by

$$H = \frac{1}{2} x' \tilde{Q} x + \frac{1}{2} \sigma' S \sigma + \lambda_x' \tilde{A} x + \lambda_{\sigma}' \tilde{A} \sigma - \lambda_{\sigma}' B \tilde{A} x \quad (3.15)$$

$$\dot{\lambda}_x = - \frac{\partial H}{\partial x} = -(\tilde{Q} x + \tilde{A}' \lambda_x - \tilde{A}' B' \lambda_{\sigma}) \quad (3.16)$$

$$\dot{\lambda}_{\sigma} = - \frac{\partial H}{\partial \sigma} = -(S \sigma + \tilde{A}' \lambda_{\sigma}) \quad (3.17)$$

$\lambda_x$  and  $\lambda_{\sigma}$  are related to  $x$  and  $\sigma$  by the equations (2.17a-b). Hence we obtain the equations for  $K_{11}$ ,  $K_{12}$ ,  $K_{21}$  and  $K_{22}$  stated above. To apply the necessary condition (2.20a) we evaluate the partial derivative of  $H$  with respect to  $D$ .

$$\begin{aligned} \frac{\partial H}{\partial D} = & R D x x' + C' K_{11} x x' + C' K_{12} \sigma x' - C' B' K_{21} x x' \\ & - C' B' K_{22} \sigma x' + C' K_{21} x \sigma' + C' K_{22} \sigma \sigma' \end{aligned} \quad (3.18)$$

Hence we obtain the above equation for D along with the equations of M,  $G_1$  and  $G_2$ .

### 3.2. Output Feedback Control

Referring to the general problem formulation stated in Section 2.1, we have, in this case, a state equation (2.7) along with a control law (2.5) and a performance index (2.14).

The equations to be solved are summarized as follows:

$$K_{11} \tilde{A} + \tilde{A}' K_{11} = -\tilde{Q} + K_{12} \tilde{B} \tilde{A} + \tilde{A}' B' K_{21} \quad (3.19)$$

$$K_{12} \tilde{A} + \tilde{A}' K_{12} = \tilde{A}' B' K_{22} \quad (3.20)$$

$$K_{21} \tilde{A} + \tilde{A}' K_{21} = K_{22} \tilde{B} \tilde{A} \quad (3.21)$$

$$K_{22} \tilde{A} + \tilde{A}' K_{22} = -S \quad (3.22)$$

$$\tilde{A} M + M \tilde{A}' = -\Sigma \quad (3.23)$$

$$\tilde{A} G_1 + G_1 \tilde{A}' = \tilde{B} \tilde{A} M \quad (3.24)$$

$$\tilde{A} G_2 + G_2 \tilde{A}' = G_1 \tilde{A}' B' + \tilde{B} \tilde{A} G_1' \quad (3.25)$$

$$\begin{aligned} D = R^{-1} C' \left\{ \begin{aligned} & (B' K_{21} - K_{11}) M + (B' K_{22} - K_{12}) G_1 \\ & - K_{21} G_1' - K_{22} G_2' \end{aligned} \right\} E' (E M E')^{-1} \end{aligned} \quad (3.26)$$

where

$$\tilde{A} = A + B + CDE \quad (3.27)$$

$$\tilde{Q} = Q + E'D'RDE \quad (3.28)$$

and

$$\Sigma = E\{x_o \ x_o'\} \quad (3.29)$$

The above equations are obtained directly by applying the optimization procedure described in Section 2.1. Here we present the main steps of the derivation. For  $\mu = 0$ , we have

$$\dot{x} = \tilde{A}x \quad (3.30)$$

$$\dot{\sigma} = \tilde{A}\sigma - B\tilde{A}x \quad (3.31)$$

and

$$J = \frac{1}{2} \int_0^\infty (x'\tilde{Q}x + \sigma'S\sigma)dt \quad (3.32)$$

where  $\tilde{A}$  and  $\tilde{Q}$  are defined above. The Hamiltonian is given by

$$H = \frac{1}{2}x'\tilde{Q}x + \frac{1}{2}\sigma'S\sigma + \lambda_x'\tilde{A}x + \lambda_\sigma'\tilde{A}\sigma - \lambda_\sigma'B\tilde{A}x \quad (3.33)$$

We notice that  $H$ , as a function of  $\tilde{A}$  and  $\tilde{Q}$ , has the same form as that at Section 3.1. Hence the equations for  $K_{11}$ ,  $K_{12}$ ,  $K_{21}$  and  $K_{22}$  are of the same form as those of Section 3.1. The partial derivative of  $H$  with respect to  $D$  is then evaluated and the necessary condition (2.20a) gives the above equation along with the definitions of  $M$ ,  $G_1$  and  $G_2$ .

### 3.3. Use of Observers

Referring to the general problem formulation stated in Section 2.1, we have, in this case, a state equation (2.7), an output equation (2.2) and an observer (2.3) along with a control law (2.6) and a performance index (2.14).

The equations to be solved are summarized as follows:

$$K_{11}\tilde{A} + \tilde{A}'K_{11} = -Q - E'K'RKE - E'G'K_{21} + \tilde{A}'B'K_{31} + K_{13}B\tilde{A} - K_{12}GE \quad (3.34)$$

$$K_{12}F + \tilde{A}'K_{12} = -E'K'RD - E'G'K_{22} + \tilde{A}'B'K_{32} + K_{13}BCD - K_{11}CD \quad (3.35)$$

$$K_{13}\tilde{A} + \tilde{A}'K_{13} = -E'G'K_{23} + \tilde{A}'B'K_{33} - K_{14}GE \quad (3.36)$$

$$K_{14}F + \tilde{A}'K_{14} = -E'G'K_{24} + \tilde{A}'B'K_{34} - K_{13}CD \quad (3.37)$$

$$K_{21}\tilde{A} + F'K_{21} = -D'RKE - D'C'K_{11} + D'C'B'K_{31} + K_{23}B\tilde{A} - K_{22}GE \quad (3.38)$$

$$K_{22}F + F'K_{22} = -D'RD - D'C'K_{12} + D'C'B'K_{32} + K_{23}BCD - K_{21}CD \quad (3.39)$$

$$K_{23}\tilde{A} + F'K_{23} = -D'C'K_{13} + D'C'B'K_{33} - K_{24}GE \quad (3.40)$$

$$K_{24}F + F'K_{24} = -D'C'K_{14} + D'C'B'K_{34} - K_{23}CD \quad (3.41)$$

$$K_{31}\tilde{A} + \tilde{A}'K_{31} = -E'G'K_{41} + K_{33}B\tilde{A} - K_{32}GE \quad (3.42)$$

$$K_{32}F + \tilde{A}'K_{32} = -E'G'K_{42} + K_{33}BCD - K_{31}CD \quad (3.43)$$

$$K_{33}\tilde{A} + \tilde{A}'K_{33} = -S - E'G'K_{43} - K_{34}GE \quad (3.44)$$

$$K_{34}F + \tilde{A}'K_{34} = -E'G'K_{44} - K_{33}CD \quad (3.45)$$

$$K_{41}\tilde{A} + F'K_{41} = -D'C'K_{31} + K_{43}B\tilde{A} - K_{42}GE \quad (3.46)$$

$$K_{42}F + F'K_{42} = -D'C'K_{32} + K_{43}BCD - K_{41}CD \quad (3.47)$$

$$K_{43}\tilde{A} + F'K_{43} = -D'C'K_{33} - K_{44}GE \quad (3.48)$$

$$K_{44}F + F'K_{44} = -D'C'K_{34} - K_{43}CD \quad (3.49)$$

$$D = R^{-1}\{C'[B'P_2 - P_1 - P_3] - RKEM_{12}\}M_{22}^{-1} \quad (3.50)$$

$$K = R^{-1}\{C'[B'P_5 - P_6 - P_4] - RDM_{21}\}E'(EM_{11}E')^{-1} \quad (3.51)$$

where

$$\tilde{A} = A + B + CKE \quad (3.52)$$

$$P_1 = K_{11}M_{12} + K_{12}M_{22} + K_{13}M_{32} + K_{14}M_{42} \quad (3.53)$$

$$P_2 = K_{31}M_{12} + K_{32}M_{22} + K_{33}M_{32} + K_{34}M_{42} \quad (3.54)$$

$$P_3 = K_{31}M_{14} + K_{32}M_{24} + K_{33}M_{34} + K_{34}M_{44} \quad (3.55)$$

$$P_4 = K_{11}M_{11} + K_{12}M_{21} + K_{13}M_{31} + K_{14}M_{41} \quad (3.56)$$

$$P_5 = K_{31}M_{11} + K_{32}M_{21} + K_{33}M_{31} + K_{34}M_{41} \quad (3.57)$$

$$P_6 = K_{31}M_{13} + K_{32}M_{23} + K_{33}M_{33} + K_{34}M_{43} \quad (3.58)$$

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \quad (3.59)$$

$$\bar{A}M + M\bar{A}' + \bar{\Sigma} = 0 \quad (3.60)$$

$$\bar{A} = \begin{bmatrix} \tilde{A} & CD & 0 & 0 \\ GE & F & 0 & 0 \\ -B\tilde{A} & -BCD & \tilde{A} & CD \\ 0 & 0 & GE & F \end{bmatrix} \quad (3.61)$$

$$\bar{\Sigma} = \begin{bmatrix} \Sigma_x & 0 & 0 & 0 \\ 0 & \Sigma_z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.62)$$

$$\Sigma_x = E\{x_o \ x_o'\} \quad (3.63)$$

$$\Sigma_z = E\{z_o \ z_o'\} \quad (3.64)$$

Here we present the main steps of the derivation of the above equations. For  $\mu = 0$ , we have

$$\dot{x} = \tilde{A}x + CDz \quad (3.65)$$

$$\dot{z} = GEx + Fz \quad (3.66)$$

$$\dot{\sigma}_x = -B\tilde{A}x - BCDz + \tilde{A}\sigma_x + CD\sigma_z \quad (3.67)$$

$$\dot{\sigma}_z = GE\sigma_x + F\sigma_z \quad (3.68)$$

The Hamiltonian is given by

$$\begin{aligned}
H = & \frac{1}{2} x' Q x + \frac{1}{2} z' D' R D z + \frac{1}{2} x' E' K' R K E x + \frac{1}{2} x' E' K' R D z + \frac{1}{2} z' D' R K E x \\
& + \frac{1}{2} \sigma_x' S \sigma_x + \lambda_x' \tilde{A} x + \lambda_x' C D z + \lambda_z' G E x + \lambda_z' F z + \lambda_{\sigma_x}' [-\tilde{B} \tilde{A} x - B C D z + \tilde{A} \sigma_x + C D \sigma_z] \\
& + \lambda_{\sigma_z}' [G E \sigma_x + F \sigma_z] \quad (3.69)
\end{aligned}$$

Then, the equations summarized above follow from the optimization procedure described in Section 2.1 along the proper definitions of  $\tilde{A}$ ,  $\bar{A}$  and  $\bar{\Sigma}$ .

## CHAPTER 4

## DETERMINISTIC SYSTEMS WITH A TIME DELAY IN THE FEEDBACK PATH

4.1. State Feedback Control

In this case, we have a state equation (2.1) along with a control law (2.8) and a performance index (2.14).

The equations to be solved are summarized as follows:

$$K_{11}\tilde{A} + \tilde{A}'K_{11} = -\tilde{Q} + K_{12}CD\tilde{A} + \tilde{A}'D'C'K_{21} \quad (4.1)$$

$$K_{12}\tilde{A} + \tilde{A}'K_{12} = \tilde{A}'D'C'K_{22} \quad (4.2)$$

$$K_{21}\tilde{A} + \tilde{A}'K_{21} = K_{22}CD\tilde{A} \quad (4.3)$$

$$K_{22}\tilde{A} + \tilde{A}'K_{22} = -S \quad (4.4)$$

$$\tilde{A}M + M\tilde{A}' = -\Sigma \quad (4.5)$$

$$\tilde{A}G_1 + G_1\tilde{A}' = CD\tilde{A}M \quad (4.6)$$

$$\tilde{A}G_2 + G_2\tilde{A}' = G_1\tilde{A}'D'C' + CD\tilde{A}G_1' \quad (4.7)$$

$$D = R^{-1}C'\{(D'C'K_{21} - K_{11})M + (D'C'K_{22} - K_{12})G_1 + (K_{21}M + K_{22}G_1')D'C' - K_{21}G_1' - K_{22}G_2'\}M^{-1} \quad (4.8)$$

where

$$\tilde{A} = A + CD \quad (4.9)$$

$$\tilde{Q} = Q + D'RD \quad (4.10)$$

and



$$\Sigma = E \begin{bmatrix} x_0 & x_0' \end{bmatrix} \quad (4.11)$$

The above equations are obtained directly by applying the optimization procedure described in Section 2.1. Here we present the main steps of the derivation. For  $\mu = 0$ , we have

$$\dot{x} = \tilde{A}x \quad (4.12)$$

$$\dot{\sigma} = \tilde{A}\sigma - CD\tilde{A}x \quad (4.13)$$

and

$$J = \frac{1}{2} \int_0^\infty (x' \tilde{Q}x + \sigma' S \sigma) dt \quad (4.14)$$

where  $\tilde{A}$  and  $\tilde{Q}$  are defined above. The Hamiltonian is given by

$$H = \frac{1}{2} x' \tilde{Q}x + \frac{1}{2} \sigma' S \sigma + \lambda_x' \tilde{A}x + \lambda_\sigma' \tilde{A}\sigma - \lambda_\sigma' CD\tilde{A}x \quad (4.15)$$

Then, the equations summarized above follow from the optimization procedure described in Section 2.1 along with the proper definitions of  $M$ ,  $G_1$  and  $G_2$ .

#### 4.2. Output Feedback Control

Referring to Section 2.1, we have, in this case, a state equation (2.1) along with a control law (2.9) and a performance index (2.14).

The equations to be solved are summarized as follows:

$$K_{11}\tilde{A} + \tilde{A}'K_{11} = -\tilde{Q} + K_{12}CDE\tilde{A} + \tilde{A}'E'D'C'K_{21} \quad (4.16)$$

$$K_{12}\tilde{A} + \tilde{A}'K_{12} = \tilde{A}'E'D'C'K_{22} \quad (4.17)$$

$$K_{21}\tilde{A} + \tilde{A}'K_{21} = K_{22}CDE\tilde{A} \quad (4.18)$$

$$K_{22}\tilde{A} + \tilde{A}'K_{22} = -S \quad (4.19)$$

$$\tilde{A}M + M\tilde{A}' = -\Sigma \quad (4.20)$$

$$\tilde{A}G_1 + G_1\tilde{A}' = CDE\tilde{A}M \quad (4.21)$$

$$\tilde{A}G_2 + G_2\tilde{A}' = G_1\tilde{A}'E'D'C' + CDE\tilde{A}G_1' \quad (4.22)$$

$$D = R^{-1}C' \left\{ (E'D'C'K_{21} - K_{11})M + (E'D'C'K_{22} - K_{12})G_1 + (K_{21}M + K_{22}G_1)E'D'C' - K_{21}G_1' - K_{22}G_2' \right\} E'(EME')^{-1} \quad (4.23)$$

where

$$\tilde{A} = A + CDE \quad (4.24)$$

$$\tilde{Q} = Q + E'D'RDE \quad (4.25)$$

and

$$\Sigma = E\{x_o \ x_o'\} \quad (4.26)$$

The above equations are obtained directly by applying the optimization procedure described in Section 2.1. Here we present the main steps of the derivation. For  $\mu = 0$ , we have

$$\dot{x} = \tilde{A}x \quad (4.27)$$

$$\dot{\sigma} = \tilde{A}\sigma - CDE\tilde{A}x \quad (4.28)$$

and

$$J = \frac{1}{2} \int_0^{\infty} (x' \tilde{Q} x + \sigma' S \sigma) dt \quad (4.29)$$

where  $\tilde{A}$  and  $\tilde{Q}$  are defined above. The Hamiltonian is given by

$$H = \frac{1}{2} x' \tilde{Q} x + \frac{1}{2} \sigma' S \sigma + \lambda_x' \tilde{A} x + \lambda_{\sigma}' \tilde{A} \sigma - \lambda_{\sigma}' C D E \tilde{A} x \quad (4.30)$$

Then, the equations summarized above follow from the optimization procedure described in Section 2.1 along with the proper definitions of  $M$ ,  $G_1$  and  $G_2$ .

#### 4.3. Use of Observers with the Delay in the Reconstructed State

Referring to Section 2.1, we have, in this case, a state equation (2.1), an output equation (2.2) and an observer (2.3) along with a control law (2.10) and a performance index (2.14).

The equations to be solved are summarized as follows:

$$K_{11} \tilde{A} + \tilde{A}' K_{11} = -Q - E' K' R K E - E' G' K_{21} + E' G' D' C' K_{31} + K_{13} C D G E - K_{12} G E \quad (4.31)$$

$$K_{12} F + \tilde{A}' K_{12} = -E' K' R D - E' G' K_{22} + E' G' D' C' K_{32} + K_{13} C D F - K_{11} C D \quad (4.32)$$

$$K_{13} \tilde{A} + \tilde{A}' K_{13} = -E' G' K_{23} + E' G' D' C' K_{33} - K_{14} G E \quad (4.33)$$

$$K_{14} F + \tilde{A}' K_{14} = -E' G' K_{24} + E' G' D' C' K_{34} - K_{13} C D \quad (4.34)$$

$$K_{21} \tilde{A} + F' K_{21} = -D' R K E - D' C' K_{11} + F' D' C' K_{31} + K_{23} C D G E - K_{22} G E \quad (4.35)$$

$$K_{22} F + F' K_{22} = -D' R D - D' C' K_{12} + F' D' C' K_{32} + K_{23} C D F - K_{21} C D \quad (4.36)$$

$$K_{23} \tilde{A} + F' K_{23} = -D' C' K_{13} + F' D' C' K_{33} - K_{24} G E \quad (4.37)$$

$$K_{24} F + F' K_{24} = -D' C' K_{14} + F' D' C' K_{34} - K_{23} C D \quad (4.38)$$

$$K_{31} \tilde{A} + \tilde{A}' K_{31} = -E' G' K_{41} - K_{32} G E + K_{33} C D G E \quad (4.39)$$

$$K_{32}F + \tilde{A}'K_{32} = -E'G'K_{42} + K_{33}CDF - K_{31}CD \quad (4.40)$$

$$K_{33}\tilde{A} + \tilde{A}'K_{33} = -S - E'G'K_{43} - K_{34}GE \quad (4.41)$$

$$K_{34}F + \tilde{A}'K_{34} = -E'G'K_{44} - K_{33}CD \quad (4.42)$$

$$K_{41}\tilde{A} + F'K_{41} = -D'C'K_{31} + K_{43}CDGE - K_{42}GE \quad (4.43)$$

$$K_{42}F + F'K_{42} = -D'C'K_{32} + K_{43}CDF - K_{41}CD \quad (4.44)$$

$$K_{43}\tilde{A} + F'K_{43} = -D'C'K_{33} - K_{44}GE \quad (4.45)$$

$$K_{44}F + F'K_{44} = -D'C'K_{34} - K_{43}CD \quad (4.46)$$

$$D = R^{-1} \left\{ C' [P_2 E' G' + P_3 F' - P_1 - P_4] - RKEM_{12} \right\} M_{22}^{-1} \quad (4.47)$$

$$K = R^{-1} \left\{ -[RDM_{21} + C'(P_5 + P_6)]E' \right\} (EM_{11}E')^{-1} \quad (4.48)$$

where

$$\tilde{A} = A + CKE \quad (4.49)$$

$$P_1 = K_{11}M_{12} + K_{12}M_{22} + K_{13}M_{32} + K_{14}M_{42} \quad (4.50)$$

$$P_2 = K_{31}M_{11} + K_{32}M_{21} + K_{33}M_{31} + K_{34}M_{41} \quad (4.51)$$

$$P_3 = K_{31}M_{12} + K_{32}M_{22} + K_{33}M_{32} + K_{34}M_{42} \quad (4.52)$$

$$P_4 = K_{31}M_{14} + K_{32}M_{24} + K_{33}M_{34} + K_{34}M_{44} \quad (4.53)$$

$$P_5 = K_{11}M_{11} + K_{12}M_{21} + K_{13}M_{31} + K_{14}M_{41} \quad (4.54)$$

$$P_6 = K_{31}M_{13} + K_{32}M_{23} + K_{33}M_{33} + K_{34}M_{43} \quad (4.55)$$

$$\bar{A}M + M\bar{A}' + \bar{\Sigma} = 0 \quad (4.56)$$

$$\bar{A} = \begin{bmatrix} \tilde{A} & CD & 0 & 0 \\ GE & F & 0 & 0 \\ -CDGE & -CDF & \tilde{A} & CD \\ 0 & 0 & GE & F \end{bmatrix} \quad (4.57)$$

$M_{ij}, i, j=1,2,3,4$  are the partitions of the matrix  $M$  as in equation (3.59) and  $\bar{\Sigma}$  is defined by equations (3.62)-(3.64).

Here we present the main steps of the derivation of the above equations. For  $\mu = 0$ , we have

$$\dot{x} = \tilde{A}x + CDz \quad (4.58)$$

$$\dot{z} = GEx + Fz \quad (4.59)$$

$$\dot{\sigma}_x = -CDGE x - CDFz + \tilde{A}\sigma_x + CD\sigma_z \quad (4.60)$$

$$\dot{\sigma}_z = GE\sigma_x + F\sigma_z \quad (4.61)$$

The Hamiltonian is given by

$$\begin{aligned} H = & \frac{1}{2} x' Q x + \frac{1}{2} z' D' R D z + \frac{1}{2} x' E' K' R K E x + \frac{1}{2} x' E' K' R D z + \frac{1}{2} z' D' R K E x + \frac{1}{2} \sigma_x' S \sigma_x \\ & + \lambda_x' (\tilde{A}x + CDz) + \lambda_z' (GE x + Fz) + \lambda_{\sigma_x}' (-CDGE x - CDFz + \tilde{A}\sigma_x + CD\sigma_z) + \lambda_{\sigma_z}' (GE\sigma_x + F\sigma_z) \end{aligned} \quad (5.80)$$

Then, the equations summarized above follow from the optimization procedure described in Section 2.1 along with the proper definitions of  $\tilde{A}$ ,  $\bar{A}$  and  $\bar{\Sigma}$ .

#### 4.4. Use of Observers with the Delay in the System Output

Referring to Section 2.1, we have, in this case, a state equation (2.1), an output equation (2.2) and an observer (2.3) along with a control law (2.11) and a performance index (2.14).

The equations to be solved are summarized as follows:

$$\begin{aligned} K_{11}\tilde{A} + \tilde{A}'K_{11} = & -Q - E'K'RKE - E'G'K_{21} + \tilde{A}'E'K'C'K_{31} + \tilde{A}'E'G'K_{41} + K_{14}GE\tilde{A} \\ & + K_{13}CKE\tilde{A} \end{aligned} \quad (4.62)$$

$$\begin{aligned} K_{12}F + \tilde{A}'K_{12} = & -E'K'RD - E'G'K_{22} + \tilde{A}'E'K'C'K_{32} + \tilde{A}'E'G'K_{42} + K_{14}GECD \\ & + K_{13}CKECD - K_{11}CD \end{aligned} \quad (4.63)$$

$$K_{13}\tilde{A} + \tilde{A}'K_{13} = -E'G'K_{23} + \tilde{A}'E'K'C'K_{33} + \tilde{A}'E'G'K_{43} - K_{14}GE \quad (4.64)$$

$$K_{14}F + \tilde{A}'K_{14} = -E'G'K_{24} + \tilde{A}'E'K'C'K_{34} + \tilde{A}'E'G'K_{44} - K_{13}CD \quad (4.65)$$

$$\begin{aligned} K_{21}\tilde{A} + F'K_{21} = & -D'RKE - D'C'K_{11} + D'C'E'K'C'K_{31} + D'C'E'G'K_{41} + K_{24}GE\tilde{A} \\ & + K_{23}CKE\tilde{A} - K_{22}GE \end{aligned} \quad (4.66)$$

$$\begin{aligned} K_{22}F + F'K_{22} = & -D'RD - D'C'K_{12} + D'C'E'K'C'K_{32} + D'C'E'G'K_{42} + K_{24}GECD \\ & + K_{23}CKECD - K_{21}CD \end{aligned} \quad (4.67)$$

$$K_{23}\tilde{A} + F'K_{23} = -D'C'K_{13} + D'C'E'K'C'K_{33} + D'C'E'G'K_{43} - K_{24}GE \quad (4.68)$$

$$K_{24}F + F'K_{24} = -D'C'K_{14} + D'C'E'K'C'K_{34} + D'C'E'G'K_{44} - K_{23}CD \quad (4.69)$$

$$K_{31}\tilde{A} + \tilde{A}'K_{31} = -E'G'K_{41} + K_{43}GE\tilde{A} + K_{33}CKE\tilde{A} - K_{32}GE \quad (4.70)$$

$$K_{32}F + \tilde{A}'K_{32} = -E'G'K_{42} + K_{34}GECD + K_{33}CKECD - K_{31}CD \quad (4.71)$$

$$K_{33}\tilde{A} + \tilde{A}'K_{33} = -S - E'G'K_{43} - K_{34}GE \quad (4.72)$$

$$K_{34}F + \tilde{A}'K_{34} = -E'G'K_{44} - K_{33}CD \quad (4.73)$$

$$K_{41}\tilde{A} + F'K_{41} = -D'C'K_{31} + K_{44}GE\tilde{A} + K_{43}CKE\tilde{A} - K_{42}GE \quad (4.74)$$

$$K_{42}F + F'K_{42} = -\tilde{D}'C'K_{32} + K_{44}GECD + K_{43}CKECD - K_{41}CD \quad (4.75)$$

$$K_{43}\tilde{A} + F'K_{43} = -D'C'K_{33} - K_{44}GE \quad (4.76)$$

$$K_{44}F + F'K_{44} = -D'C'K_{34} - K_{43}CD \quad (4.77)$$

$$D = R^{-1} \left\{ C'[E'K'C'P_2 + E'G'P_4 - P_1 - P_3] - RKEM_{12} \right\} M_{22}^{-1} \quad (4.78)$$

$$K = R^{-1} \left\{ C'[P_6E'K'C' + E'K'C'P_6 + P_2D'C' + E'G'P_8 - P_5 - P_7] - RDM_{21} \right\} E'(EM_{11}E')^{-1} \quad (4.79)$$

where

$$\tilde{A} = A + CKE \quad (4.80)$$

$$P_1 = K_{11}M_{12} + K_{12}M_{22} + K_{13}M_{32} + K_{14}M_{42} \quad (4.81)$$

$$P_2 = K_{31}M_{12} + K_{32}M_{22} + K_{33}M_{32} + K_{34}M_{42} \quad (4.82)$$

$$P_3 = K_{31}M_{14} + K_{32}M_{24} + K_{33}M_{34} + K_{34}M_{44} \quad (4.83)$$

$$P_4 = K_{41}M_{12} + K_{42}M_{22} + K_{43}M_{32} + K_{44}M_{42} \quad (4.84)$$

$$P_5 = K_{11}M_{11} + K_{12}M_{21} + K_{13}M_{31} + K_{14}M_{41} \quad (4.85)$$

$$P_6 = K_{31}M_{11} + K_{32}M_{21} + K_{33}M_{31} + K_{34}M_{41} \quad (4.86)$$

$$P_7 = K_{31}M_{13} + K_{32}M_{23} + K_{33}M_{33} + K_{34}M_{43} \quad (4.87)$$

$$P_8 = K_{41}M_{11} + K_{42}M_{21} + K_{43}M_{31} + K_{44}M_{41} \quad (4.88)$$

$$\bar{A}M + M\bar{A}' + \bar{\Sigma} = 0 \quad (4.89)$$

$$\bar{A} = \begin{bmatrix} \tilde{A} & CD & 0 & 0 \\ GE & F & 0 & 0 \\ -CKE\tilde{A} & -CKECD & \tilde{A} & CD \\ -GE\tilde{A} & -GECD & GE & F \end{bmatrix} \quad (4.90)$$

$M_{ij}$ ,  $i, j=1,2,3,4$  are partitions of  $M$  as in equation (3.59) and  $\bar{\Sigma}$  is defined by (3.62) - (3.64).

Here we present the main steps of the derivation of the above equations. For  $\mu = 0$ , we have

$$\dot{x} = \tilde{A}x + CDz \quad (4.91)$$

$$\dot{z} = GEx + Fz \quad (4.92)$$

$$\dot{\sigma}_x = -CKE\tilde{A}x - CKECDz + \tilde{A}\sigma_x + CD\sigma_z \quad (4.93)$$

$$\dot{\sigma}_z = -GE\tilde{A}x - GECDz + GE\sigma_x + F\sigma_z \quad (4.94)$$

The Hamiltonian is given by

$$\begin{aligned} H = & \frac{1}{2} x' Qx + \frac{1}{2} z' D' R D z + \frac{1}{2} x' E' K' R K E x + \frac{1}{2} x' E' K' R D z + \frac{1}{2} z' D' R K E x + \frac{1}{2} \sigma_x' S \sigma_x \\ & + \lambda_x' (\tilde{A}x + CDz) + \lambda_z' (GEx + Fz) + \lambda_{\sigma_x}' (-CKE\tilde{A}x - CKECDz + \tilde{A}\sigma_x + CD\sigma_z) \\ & + \lambda_{\sigma_z}' (-GE\tilde{A}x - GECDz + GE\sigma_x + F\sigma_z) \end{aligned} \quad (4.95)$$

Then, the equations summarized above follow from the optimization procedure described in Section 2.1 along with the proper definitions of  $\tilde{A}$ ,  $\bar{A}$  and  $\bar{\Sigma}$ .



## CHAPTER 5

## STOCHASTIC SYSTEMS WITH A TIME DELAY IN THE PLANT

5.1. State Feedback Control

Referring to section 2.2, we have, in this case, a state equation (2.27) along with a control law (2.4) and a performance index (2.33).

The equations to be solved are summarized as follows:

$$\bar{A}'K + K\bar{A} + \bar{Q} = 0 \quad (5.1)$$

$$\bar{A}P + P\bar{A}' + \bar{\Sigma} = 0 \quad (5.2)$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad (5.3) \quad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \quad (5.4)$$

$$KP = \begin{bmatrix} (KP)_{11} & (KP)_{12} \\ (KP)_{21} & (KP)_{22} \end{bmatrix} \quad (5.5)$$

$$D = R^{-1} \{ C' [ B' (KP)_{21} - (KP)_{11} - (KP)_{22} ] \} \cdot P_{11}^{-1} \quad (5.6)$$

where

$$\tilde{A} = A + B + CD \quad (5.7)$$

$$\tilde{Q} = Q + D'RD \quad (5.8)$$

$$\Sigma = E\{w_1 w_1'\} \quad (5.9)$$

$$\bar{A} = \begin{bmatrix} \tilde{A} & 0 \\ -B\tilde{A} & \tilde{A} \end{bmatrix} \quad (5.10)$$

$$\bar{Q} = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & S \end{bmatrix} \quad (5.11)$$

$$\bar{\Sigma} = \begin{bmatrix} \Sigma & -\Sigma B' \\ -B\Sigma & B\Sigma B' \end{bmatrix} \quad (5.12)$$

The above equations are obtained directly by applying the optimization procedure described in section 2.2. Here we present the main steps of the derivation. For  $\mu = 0$ , we have

$$\dot{x} = \tilde{A}x + w_1 \quad (5.13)$$

$$\dot{\sigma} = \tilde{A}\sigma - B\tilde{A}x - Bw_1 \quad (5.14)$$

Define

$$\bar{x} = \begin{bmatrix} x \\ \sigma \end{bmatrix} \quad (5.15) ,$$

$$\bar{w} = \begin{bmatrix} w_1 \\ -Bw_1 \end{bmatrix} \quad (5.16)$$

Hence, we have an equivalent static minimization problem described by equations (2.39)-(2.40) with  $\bar{A}$ ,  $\bar{Q}$  and  $\bar{\Sigma}$  as defined above. The necessary conditions follow directly from equations (2.41)-(2.43) making use of the fact that a trace of a block matrix is equal to the sum of traces of the diagonal blocks.

## 5.2. Output Feedback Control

Referring to section 2.2, we have, in this case, a state equation (2.27) and an output equation (2.24) along with a control law (2.26) and a performance index (2.33).

The equations to be solved are summarized as follows:

$$\bar{A}'K + K\bar{A} + \bar{Q} = 0 \quad (5.17)$$

$$\bar{A}P + P\bar{A}' + \bar{\Sigma} = 0 \quad (5.18)$$

$$D = R^{-1} \begin{bmatrix} L_1 & \end{bmatrix} (EP_{11}E')^{-1} \quad (5.19)$$

$$L_1 = C'\{B'(KP)_{21} - (KP)_{11} - (KP)_{22}\}E' \quad (5.20)$$

where

$$\tilde{A} = A + B + CDE \quad (5.21)$$

$$\tilde{Q} = Q + E'D'RDE \quad (5.22)$$

$$\Sigma_1 = E\{w_1 w_1'\} \quad (5.23)$$

$$\bar{A} = \begin{bmatrix} \tilde{A} & 0 \\ -B\tilde{A} & \tilde{A} \end{bmatrix}, \quad (5.24) \quad \bar{Q} = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & S \end{bmatrix} \quad (5.25)$$

$$\bar{\Sigma} = \begin{bmatrix} \Sigma_1 & -\Sigma_1 B' \\ -B\Sigma_1 & B\Sigma_1 B' \end{bmatrix} \quad (5.26)$$

$P_{ij}$ ,  $K_{ij}$  and  $(KP)_{ij}$ ;  $i, j = 1, 2$  are partitions of  $P$ ,  $K$  and  $(KP)$  as in equation (5.3) - (5.5).

The above equations are obtained directly by applying the optimization procedure described in section 2.2. Here we present the

main steps of the derivation. For  $u = 0$ , we have

$$\dot{x} = \tilde{A}x + w_1 \quad (5.27)$$

$$\dot{\sigma} = \tilde{A}\sigma - B\tilde{A}x - Bw_1 \quad (5.28)$$

Define

$$\bar{x} = \begin{bmatrix} x \\ \sigma \end{bmatrix} \quad (5.29) \quad , \quad \bar{w} = \begin{bmatrix} w_1 \\ -Bw_1 \end{bmatrix} \quad (5.30)$$

Hence, we have an equivalent static minimization problem described by equations (2.39)-(2.40) with  $\bar{A}$ ,  $\bar{Q}$  and  $\bar{\Sigma}$  as defined above. The necessary conditions follow directly from equations (2.41)-(2.43).

### 5.3. Use of Observers

Referring to section 2.2, we have, in this case, a state equation (2.27), an output equation (2.24) and an observer (2.25) along with a performance index (2.33).

The equations to be solved are summarized as follows

$$\bar{A}'K + K\bar{A} + \bar{Q} = 0 \quad (5.31)$$

$$\bar{A}P + P\bar{A}' + \bar{\Sigma} = 0 \quad (5.32)$$

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix} \quad (5.33)$$

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \quad (5.34)$$

$$KP = \begin{bmatrix} (KP)_{11} & (KP)_{12} & (KP)_{13} & (KP)_{14} \\ (KP)_{21} & (KP)_{22} & (KP)_{23} & (KP)_{24} \\ (KP)_{31} & (KP)_{32} & (KP)_{33} & (KP)_{34} \\ (KP)_{41} & (KP)_{42} & (KP)_{43} & (KP)_{44} \end{bmatrix} \quad (5.35)$$

$$D = R^{-1} \{ C' [ B' (KP)_{32} - (KP)_{12} - (KP)_{34} - (KP)_{22} - (KP)_{44} ] \} P_{22}^{-1} \quad (5.36)$$

$$F = K_{22}^{-1} \{ [ (KP)_{22} + (KP)_{44} - (KP)_{21} - (KP)_{43} ] E' \} \Sigma_2^{-1} \quad (5.37)$$

where

$$\tilde{A} = A + B + CD - FE \quad (5.38)$$

$$\Sigma_1 = E \{ w_1 w_1' \} \quad (5.39)$$

$$\Sigma_2 = E \{ w_2 w_2' \} \quad (5.40)$$

$$\bar{A} = \begin{bmatrix} A+B & CD & 0 & 0 \\ FE & \tilde{A} & 0 & 0 \\ -B(A+B) & -BCD & A+B & CD \\ 0 & 0 & FE & \tilde{A} \end{bmatrix} \quad (5.41)$$

$$\bar{Q} = \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & D'RD & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.42)$$

$$\bar{\Sigma} = \begin{bmatrix} \Sigma_1 & 0 & -\Sigma_1 B' & 0 \\ 0 & F\Sigma_2 F' & 0 & 0 \\ -B\Sigma_1 & 0 & B\Sigma_1 B' & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.43)$$

The above equations are obtained directly by applying the optimization procedure described in section 2.2. Here we present the main steps of the derivation. For  $\mu = 0$ , we have

$$\dot{x} = (A+B)x + CDz + w_1 \quad (5.44)$$

$$\dot{z} = FE\tilde{x} + \tilde{A}z + Fw_2 \quad (5.45)$$

$$\dot{\sigma}_x = -B(A+B)x - BCDz + (A+B)\sigma_x + CD\sigma_z - Bw_1 \quad (5.46)$$

$$\dot{\sigma}_z = FE\sigma_x + \tilde{A}\sigma_z \quad (5.47)$$

Define

$$\bar{x} = \begin{bmatrix} x \\ z \\ \sigma_x \\ \sigma_z \end{bmatrix} \quad (5.48)$$

$$\bar{w} = \begin{bmatrix} w_1 \\ Fw_2 \\ -Bw_1 \\ 0 \end{bmatrix} \quad (5.49)$$

Hence we have an equivalent static minimization problem described by equations (2.39)-(2.40) with  $\bar{A}$ ,  $\bar{Q}$  and  $\bar{\Sigma}$  as defined above. The necessary conditions follow directly from equations (2.41)-(2.43) with one more equation, similar to equation (2.41), for optimization of  $F$ .

## CHAPTER 6

## STOCHASTIC SYSTEMS WITH A TIME DELAY IN THE FEEDBACK PATH

6.1. State Feedback Control

Referring to section 2.2, we have, in this case, a state equation (2.23) along with a control law (2.28) and a performance index (2.33).

The equations to be solved are summarized as follows:

$$\bar{A}'K + K\bar{A} + \bar{Q} = 0 \quad (6.1)$$

$$\bar{A}P + P\bar{A}' + \bar{\Sigma} = 0 \quad (6.2)$$

$$D = R^{-1}(L_1 + L_2)P_{11}^{-1} \quad (6.3)$$

$$L_1 = C'[(KP)_{21}A' + K_{21}\Sigma_1 - (KP)_{11} - (KP)_{22}] \quad (6.4)$$

$$L_2 = C'[(KP)_{21}D'C' + D'C'(KP)_{22} - K_{22}CD\Sigma_1] \quad (6.5)$$

where

$$\tilde{A} = A + CD \quad (6.6)$$

$$\tilde{Q} = Q + D'RD \quad (6.7)$$

$$\Sigma_1 = E\{w_1 w_1'\} \quad (6.8)$$

$$\bar{A} = \begin{bmatrix} \tilde{A} & 0 \\ -CD\tilde{A} & \tilde{A} \end{bmatrix} \quad (6.9)$$

$$\bar{Q} = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & S \end{bmatrix} \quad (6.10)$$



$$\bar{\Sigma} = \begin{bmatrix} \Sigma_1 & -\Sigma_1 D' C' \\ -CD \Sigma_1 & CD \Sigma_1 D' C' \end{bmatrix} \quad (6.11)$$

$P_{ij}$ ,  $K_{ij}$  and  $(KP)_{ij}$ ;  $i, j = 1, 2$  are partitions of  $P$ ,  $K$  and  $(KP)$  as in equations (5.3)-(5.5).

The above equations are obtained directly by applying the optimization procedure described in section 2.2. Here we present the main steps of the derivation. For  $u = 0$ , we have

$$\dot{\bar{x}} = \tilde{A}x + w_1 \quad (6.12)$$

$$\dot{\bar{\sigma}} = \tilde{A}\sigma - CD\tilde{A}x - CDw_1 \quad (6.13)$$

Define

$$\bar{x} = \begin{bmatrix} x \\ \sigma \end{bmatrix}, \quad (6.14)$$

$$\bar{w} = \begin{bmatrix} w_1 \\ -CDw_1 \end{bmatrix} \quad (6.15)$$

Hence, we have an equivalent static minimization problem described by equations (2.39)-(2.40) with  $\bar{A}$ ,  $\bar{Q}$  and  $\bar{\Sigma}$  as defined above. The necessary conditions follow directly from equations (2.41)-(2.43).

## 6.2. Output Feedback Control

Referring to section 2.2, we have, in this case, a state equation (2.23) and an output equation (2.24) along with a control law (2.29) and a performance index (2.33).

The equations to be solved are summarized as follows:

$$\bar{A}'K + K\bar{A} + \bar{Q} = 0 \quad (6.16)$$

$$\bar{A}P + P\bar{A}' + \bar{\Sigma} = 0 \quad (6.17)$$

$$D = R^{-1}(L_1 + L_2)(EP_{11}E')^{-1} \quad (6.18)$$

$$L_1 = C'[(KP)_{21}A' + K_{21}\Sigma_1 - (KP)_{11} - (KP)_{22}]E' \quad (6.19)$$

$$L_2 = C'\{[(KP)_{21}E'D'C' + E'D'C'(KP)_{21} - K_{22}CDE\Sigma_1 \quad (6.20)$$

where

$$\tilde{A} = A + CDE \quad (6.21)$$

$$\tilde{Q} = Q + E'D'RDE \quad (6.22)$$

$$\Sigma_1 = E\{w_1 w_1'\} \quad (6.23)$$

$$\bar{A} = \begin{bmatrix} \tilde{A} & 0 \\ -CDE\tilde{A} & \tilde{A} \end{bmatrix} \quad (6.24)$$

$$\bar{Q} = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & S \end{bmatrix} \quad (6.25)$$

$$\bar{\Sigma} = \begin{bmatrix} \Sigma_1 & -\Sigma_1 E' D' C' \\ -CDE \Sigma_1 & CDE \Sigma_1 E' D' C' \end{bmatrix} \quad (6.26)$$

$P_{ij}$ ,  $K_{ij}$  and  $(KP)_{ij}$ ;  $i, j = 1, 2$  are partitions of  $P$ ,  $K$  and  $(KP)$  as in equations (5.3)-(5.5).

The above equations are obtained directly by applying the optimization procedure described in section 2.2. Here we present the main steps of the derivation. For  $\mu = 0$ , we have

$$\dot{\bar{x}} = \tilde{A}\bar{x} + w_1 \quad (6.27)$$

$$\dot{\bar{\sigma}} = \tilde{A}\bar{\sigma} - CDE\tilde{A}\bar{x} - CDE w_1 \quad (6.28)$$

Define

$$\bar{x} = \begin{bmatrix} x \\ \sigma \end{bmatrix}, \quad (6.29)$$

$$\bar{w} = \begin{bmatrix} w_1 \\ -CDEw_1 \end{bmatrix} \quad (6.30)$$

Hence, we have an equivalent static minimization problem described by equations (2.39)-(2.40) with  $\bar{A}$ ,  $\bar{Q}$  and  $\bar{\Sigma}$  as defined above. The necessary conditions follow directly from equations (2.41)-(2.43).

### 6.3. Use of an Observer with a Delay in the Reconstructed State

Referring to section 2.2, we have, here, a state equation (2.23), an output equation (2.24) and an observer (2.25) along with a control law (2.31) and a performance index (2.33).

The equations to be solved are summarized as follows:

$$\bar{A}'K + K\bar{A} + \bar{Q} = 0 \quad (6.31)$$

$$\bar{A}P + P\bar{A}' + \bar{\Sigma} = 0 \quad (6.32)$$

$$D = -R^{-1}C'[L_1 + L_2]P_{22}^{-1} \quad (6.33)$$

$$F = -K_{22}^{-1}[L_3 + L_4]\Sigma_2^{-1} \quad (6.34)$$

$$\begin{aligned} L_1 = & (KP)_{12} + (KP)_{22} + (KP)_{34} + (KP)_{44} - \{(KP)_{32} + (KP)_{42}\}A' \\ & - \{(KP)_{31} - (KP)_{32} + (KP)_{41} - (KP)_{42}\}E'F' - \{K_{32} + K_{42}\}F\Sigma_2F' \end{aligned} \quad (6.35)$$

$$\begin{aligned} L_2 = & \{-(KP)_{32} - (KP)_{42}\}D'C' - D'C'\{(KP)_{32} + (KP)_{42}\} \\ & + \{K_{33} + K_{34} + K_{43} + K_{44}\}CDF\Sigma_2F' \end{aligned} \quad (6.36)$$

$$\begin{aligned} L_3 = & \{(KP)_{21} - (KP)_{22} + (KP)_{43} - (KP)_{44} \\ & + D'C'\{-(KP)_{31} + (KP)_{32} - (KP)_{41} + (KP)_{42}\}E' \end{aligned} \quad (6.37)$$

$$\begin{aligned} L_4 = & [-\{K_{23} + K_{24}\}CD - D'C'(\{K_{42} + K_{32}\} \\ & - \{K_{33} + K_{43} + K_{34} + K_{44}\}CD)]F\Sigma_2 \end{aligned} \quad (6.38)$$

where

$$\tilde{A} = A + CD\_FE \quad (6.39)$$

$$\Sigma_1 = E\{w_1 w_1'\} \quad (6.40)$$

$$\Sigma_2 = E\{w_2 w_2'\} \quad (6.41)$$

$$\bar{A} = \begin{bmatrix} A & CD & 0 & 0 \\ FE & \tilde{A} & 0 & 0 \\ -CDFE & -CD\tilde{A} & A & CD \\ -CDFE & -CD\tilde{A} & FE & \tilde{A} \end{bmatrix} \quad (6.42)$$

$$\bar{Q} = \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & D'RD & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (6.43)$$

$$\bar{\Sigma} = \begin{bmatrix} \Sigma_1 & 0 & 0 & 0 \\ 0 & F\Sigma_2 F' & -F\Sigma_2 F'D'C' & -F\Sigma_2 F'D'C' \\ 0 & -CDF\Sigma_2 F' & CDF\Sigma_2 F'D'C' & CDF\Sigma_2 F'D'C' \\ 0 & -CDF\Sigma_2 F' & CDF\Sigma_2 F'D'C' & CDF\Sigma_2 F'D'C' \end{bmatrix} \quad (6.44)$$

$P_{ij}$ ,  $K_{ij}$  and  $(KP)_{ij}$ ;  $i, j=1,2,3,4$  are partitions of  $P$ ,  $K$  and  $(KP)$  as in equations (5.33) - (5.35).

The above equations are obtained directly by applying the optimization procedure described in section 2.2. Here we present the main steps of the derivation. For  $\mu = 0$ , we have

$$\dot{x} = Ax + CDz + w_1 \quad (6.45)$$

$$\dot{z} = FEx + \tilde{A}z + Fw_2 \quad (6.46)$$

$$\dot{\sigma}_x = -CDFEx - CD\tilde{A}z + A\sigma_x + CD\sigma_z - CDFw_2 \quad (6.47)$$

$$\dot{\sigma}_z = FE\sigma_x + \tilde{A}\sigma_z - CDFEx - CD\tilde{A}z - CDFw_2 \quad (6.48)$$

Define

$$\bar{x} = \begin{bmatrix} x \\ z \\ \sigma_x \\ \sigma_z \end{bmatrix} \quad (6.49)$$

$$\bar{w} = \begin{bmatrix} w_1 \\ Fw_2 \\ -CDFw_2 \\ -CDFw_2 \end{bmatrix} \quad (6.50)$$

Hence we have an equivalent static minimization problem described by equations (2.39)-(2.40) with  $\bar{A}$ ,  $\bar{Q}$  and  $\bar{\Sigma}$  as defined above. The necessary conditions follow directly from equations (2.41)-(2.43) with one more equation, similar to equation (2.41), for optimization of  $F$ .

#### 6.4. Use of an Observer with a Delay in the System Output

Referring to section 2.2, we have, in this case, a state equation (2.23), an output equation (2.30) and an observer (2.25) along with a performance index (2.33).

The equations to be solved are summarized as follows:

$$\bar{A}'K + K\bar{A} + \bar{Q} = 0 \quad (6.51)$$

$$\bar{A}P + P\bar{A}' + \bar{E} = 0 \quad (6.52)$$

$$D = R^{-1}L_1P_{22}^{-1} \quad (6.53)$$

$$F = K_{22}^{-1}(L_4 + L_5)\Sigma_2^{-1} \quad (6.54)$$

$$L_1 = C'[E'F'(KP)_{42} - (KP)_{12} - (KP)_{34} - (KP)_{22} - (KP)_{44}] \quad (6.55)$$

$$L_4 = [(KP)_{22} + (KP)_{44} - (KP)_{21} - (KP)_{43} + (KP)_{41}A' + (KP)_{42}D'C' + K_{41}\Sigma_1]E' \quad (6.56)$$

$$L_5 = -K_{44}FE\Sigma_1E' \quad (6.57)$$

where

$$\tilde{A} = A + CD - FE \quad (6.58)$$

$$\Sigma_1 = E\{w_1w_1'\} \quad (6.59)$$

$$\Sigma_2 = E\{w_2w_2'\} \quad (6.60)$$

$$\bar{A} = \begin{bmatrix} A & CD & 0 & 0 \\ FE & \tilde{A} & 0 & 0 \\ 0 & 0 & A & CD \\ -FEA & -FECD & FE & \tilde{A} \end{bmatrix} \quad (6.61)$$

$$\bar{Q} = \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & D'RD & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (6.62)$$

$$\bar{\Sigma} = \begin{bmatrix} \Sigma_1 & 0 & 0 & -\Sigma_1 E' F' \\ 0 & F \Sigma_2 F' & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -FE \Sigma_1 & 0 & 0 & FE \Sigma_1 E' F' \end{bmatrix} \quad (6.63)$$

$P_{ij}$ ,  $K_{ij}$  and  $(KP)_{ij}$ ;  $i, j = 1, 2, 3, 4$  are partitions of  $P$ ,  $K$  and  $KP$  as in equations (5.33) - (5.35).

The above equations are obtained directly by applying the optimization procedure described in section 2.2. Here we present the main steps of the derivation. For  $\mu = 0$ , we have



$$\dot{x} = Ax + CDz + w_1 \quad (6.64)$$

$$\dot{z} = FEx + \tilde{A}z + Fw_2 \quad (6.65)$$

$$\dot{\sigma}_x = A\sigma_x + CD\sigma_z \quad (6.66)$$

$$\dot{\sigma}_z = -FEAx - FEC Dz + FE\sigma_x + \tilde{A}\sigma_z - FEw_1 \quad (6.67)$$

Define

$$\bar{x} = \begin{bmatrix} x \\ z \\ \sigma_x \\ \sigma_z \end{bmatrix}, \quad (6.68)$$

$$\bar{w} = \begin{bmatrix} w_1 \\ Fw_2 \\ 0 \\ -FEw_1 \end{bmatrix} \quad (6.69)$$

Hence we have an equivalent static minimization problem described by equations (2.39)-(2.40) with  $\bar{A}$ ,  $\bar{Q}$  and  $\bar{\Sigma}$  as defined above. The necessary conditions follow directly from equations (2.41)-(2.43) with one more equation, similar to equation (2.41), for optimization of  $F$ .

## CHAPTER 7

## EXAMPLES

In this chapter we apply our low-sensitivity design scheme to two practical examples and compare the results to those of the standard LQG design.

7.1. Continuous Stirred-Tank Reactor

Consider the continuous stirred-tank reactor [25] illustrated in Figure 7.1. The problem is to control the outlet temperature,  $T$ , and the outlet concentration,  $C_A$ , at some desired set points,  $\bar{T}$  and  $\bar{C}_A$ , in such a way as to minimize a specified cost function. The manipulated variables are the feed rate,  $w$ , and the heat added or removed,  $Q$ . In practice not  $Q$  but the flow rate through a jacket or cooling coils would be manipulated. This is related to  $Q$  through an unsteady state energy balance. For this problem however it is simpler to assume  $Q$  is directly manipulated and that enough heat transfer is available so that saturation does not occur. If  $R$  is chosen relatively large this will assure that too great a control effort will not occur. The system is driven by the initial condition given for temperature and concentration. This is the standard state regulator problem and physically may be interpreted as a "start-up" problem. A second order reversible reaction is assumed. The rate of reaction is expressed by

$$\text{Rate} = \frac{dC_A}{dt} = -kC_A^2 \quad (7.1)$$

The rate constant,  $k$ , may be expressed as a function of temperature using the Arrhenius expression

$$k = k_0 \exp\left\{-\frac{a}{T}\right\} \quad (7.2)$$

Substitution of (7.2) into (7.1) yields

$$\text{Rate} = -k_0 \exp\left\{-\frac{a}{T}\right\} C_A^2 \quad (7.3)$$

Note from (7.3) that the rate of reaction is nonlinear with respect to temperature,  $T$ , and concentration in tank,  $C_A$ . The degree of nonlinearity with respect to  $T$  depends on the size of  $a$ , which is a function of the particular reaction. Unsteady state material and energy balances on the reactor yield

$$\frac{w}{v\rho}(C_{Af} - C_A) - k C_A^2 = \frac{dC_A}{dt} \quad (7.4)$$

$$\frac{w}{v\rho}(T_f - T) + \frac{Q}{v\rho C_p} - \frac{\Delta H k C_A^2}{\rho C_p} = \frac{dT}{dt} \quad (7.5)$$

Linearization of equations (7.4) and (7.5) about steady state or set-point conditions yields

$$\frac{d\hat{C}_A}{dt} = \left( -\frac{\bar{w}}{v\rho} - 2\bar{C}_A\bar{k} \right) \hat{C}_A - \left( \frac{\bar{k}a\bar{C}_A^2}{T^2} \right) \hat{T} + \left( \frac{\bar{C}_{Af} - \bar{C}_A}{v\rho} \right) \hat{w} \quad (7.6)$$

$$\frac{d\hat{T}}{dt} = \left( \frac{-2\Delta H k C_A^2}{\rho C_p} \right) \hat{C}_A - \left( \frac{\bar{w}}{v\rho} + \frac{\Delta H k C_A^2 a}{T^2} \right) \hat{T} + \frac{\hat{Q}}{v\rho C_p} + \left( \frac{\bar{T}_f - \bar{T}}{v\rho} \right) \hat{w} \quad (7.7)$$

$\hat{C}_A$  and  $\hat{T}$ ,  $\hat{Q}$ ,  $\hat{w}$  are defined as variations about the steady state values,  $\bar{C}_A$  and  $\bar{T}$ ,  $\bar{Q}$ ,  $\bar{w}$ .  $\bar{w}$  and  $\bar{k}$  are obtained from the steady state versions of (7.4) and (7.5). Defining  $\hat{T}$  and  $\hat{C}_A$  as our state variables and  $Q$  and  $w$  as manipulated variables we may write

$$\begin{aligned}
 \frac{d}{dt} \begin{bmatrix} \hat{T} \\ \hat{C}_A \end{bmatrix} &= \begin{bmatrix} \left( \frac{-\bar{w}}{v\rho} - \frac{a\Delta H\bar{k}\bar{C}_A^2}{\rho C_p \bar{T}^2} \right) & \left( \frac{-2\Delta H\bar{k}\bar{C}_A}{\rho C_p} \right) \\ \left( \frac{-a\bar{k}\bar{C}_A^2}{\bar{T}^2} \right) & \begin{pmatrix} -\frac{\bar{w}}{v\rho} & -2\bar{k}\bar{C}_A \end{pmatrix} \end{bmatrix} \begin{bmatrix} \hat{T} \\ \hat{C}_A \end{bmatrix} \\
 &+ \begin{bmatrix} \frac{1}{v\rho C_p} & \frac{\bar{T}_F - \bar{T}}{v\rho C_p} \\ 0 & \frac{\bar{C}_{AF} - \bar{C}_A}{v\rho} \end{bmatrix} \begin{bmatrix} \hat{Q} \\ \hat{W}_j \end{bmatrix} \quad (7.8)
 \end{aligned}$$

The following reactor parameters were chosen:

- $v$  = Reactor volume =  $13.38 \text{ ft}^3$
- $\rho$  = Density =  $55 \text{ lb/ft}^3$
- $C_p$  = Heat capacity =  $1.0 \text{ Btu/lb } ^\circ\text{F}$
- $\Delta H$  = Heat of reaction =  $-12,000 \text{ Btu}$
- $a$  =  $14,000 \text{ } ^\circ\text{R}$
- $k_0$  = Reaction rate constant =  $8.33 \times 10^8 \text{ ft}^3/\text{lb mole min}$
- $T_F$  = Reactor feed temperature =  $100^\circ\text{F}$
- $C_{AF}$  = Reactor feed concentration =  $0.4 \text{ lb moles/ft}^3$
- $\bar{T}$  = Reactor temperature =  $200^\circ\text{F}$
- $\bar{C}_A$  = Reactor concentration =  $0.2 \text{ lb moles/ft}^3$

Solution for the equilibrium points in (7.4) and (7.5) yields steady state flow rate, rate constant and heat duty as

$$\bar{k} = 0.51 \text{ ft}^3/\text{lb mole min}$$

$$\bar{w} = 75.2 \text{ lb/min}$$

$$\bar{Q} = 4238. \text{ Btu/min}$$

Using the above parameters our system equation (7.8) becomes

$$\dot{x} = \begin{bmatrix} 0.041 & 44.6 \\ -0.001 & -0.307 \end{bmatrix} x + \begin{bmatrix} 0.001 & -0.136 \\ 0 & 0.001 \end{bmatrix} u \quad (7.9)$$

For the state regulator design of this reactor, we should expect an undesirable time delay in measuring the system states, i.e. temperature and concentration. Such a delay might be significant enough to affect the system behavior causing deviation of the trajectories from the nominal ones, corresponding to zero delays. To reduce that effect, our proposed low-sensitivity design scheme is applied. Consider the case discussed in section 4.1 using the following design parameters

$$R = \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix} \quad (7.10)$$

$$Q = \begin{bmatrix} 1 & 10 \\ 10 & 400 \end{bmatrix} \quad (7.11)$$

$$S = 0 \quad (7.12)$$

This case corresponds to the standard state regulator problem. The feedback gain matrix is computed to be

$$D = \begin{bmatrix} -0.996 & -103 \\ 0.76 & 75.7 \end{bmatrix} \quad (7.13)$$

Trajectories of this case are plotted for several values of delays as well as for the nominal zero delay. Trajectories and controls are illustrated in Figures 7.2-7.5. It is seen that  $x_2(t)$  shows unacceptable sensitivity. So,

we choose  $S$  so as to penalize that sensitivity. Now, consider the same values of  $R$  and  $Q$  given by equations (7.10) - (7.11) along with the following values of  $S$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix} \quad (7.14)$$

The feedback gain matrix in this case is

$$D = \begin{bmatrix} -0.997 & -105.1 \\ 0.532 & -53.21 \end{bmatrix} \quad (7.15)$$

The trajectories  $x_1(t)$  and  $x_2(t)$  of this case are illustrated in Figures 7.6 and 7.7 respectively. The controls  $u_1(t)$  and  $u_2(t)$  are illustrated in Figures 7.8 and 7.9 respectively. It is seen that trajectories in this case become closer to each other than they are for the case when  $S = 0$  (Figures 7.2 and 7.3). This means that sensitivity to small undesirable delays is reduced. However, we notice that the trajectories have large overshoots. To improve this undesirable transient phenomena, we try other values of the design matrices  $Q$ ,  $R$  and  $S$ . A good choice of  $Q$ ,  $R$  and  $S$  will result in low sensitive trajectories having desirable transient phenomena.

For another set of trajectories, consider the following

$$R = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix} \quad (7.16)$$

$$Q = \begin{bmatrix} 50 & 10 \\ 10 & 400 \end{bmatrix} \quad (7.17)$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \quad (7.18)$$

The feedback gain matrix in this case is

$$D = \begin{bmatrix} -0.996 & -99 \\ 0.76 & 50 \end{bmatrix} \quad (7.19)$$

Trajectories  $x_1(t)$  and  $x_2(t)$  of this case are illustrated in Figures 7.10 and 7.11 respectively.

Now consider R and Q given by equations (7.16) and (7.17) along with the following value of S

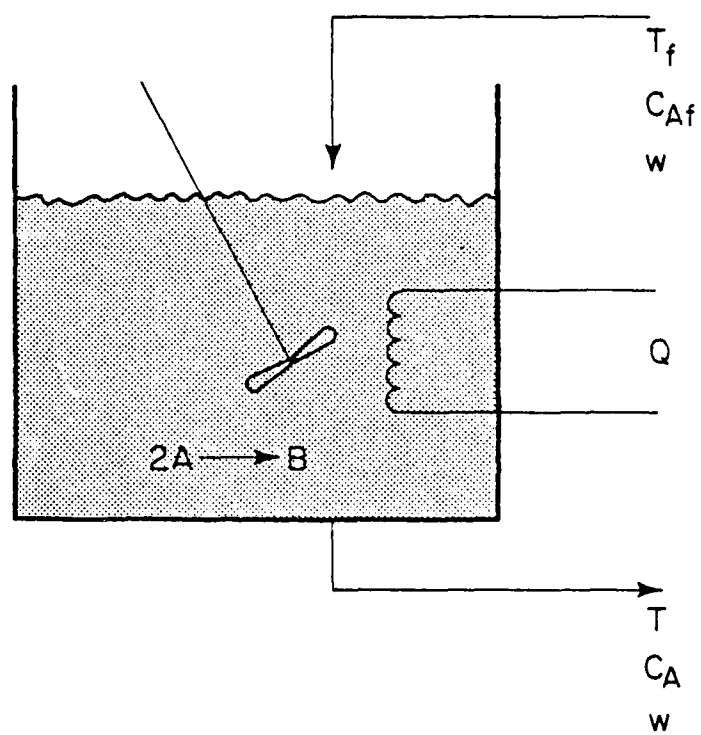
$$S = \begin{bmatrix} 500 & 0 \\ 0 & 500 \end{bmatrix} \quad (7.20)$$

The feedback gain matrix in this case is

$$D = \begin{bmatrix} -0.994 & -107 \\ 0.48 & -49.1 \end{bmatrix} \quad (7.21)$$

Trajectories and controls of this case are illustrated in Figures 7.12-7.15.

From the above discussion and associated plots we conclude that our design scheme is reliable in reducing trajectory sensitivity. This is because the matrix S has a direct handle on the trajectory sensitivity with any desired weighting. To get acceptable trajectories, several values of Q, R and S have been tried. It is seen that the choice of Q, R and S given by equations (7.17), (7.16) and (7.20) respectively has resulted in a satisfactorily low sensitivity. However, the transient response is not satisfactory. The choice given by equations (7.10), (7.11) and (7.14) showed satisfactory sensitivity as well as satisfactory transient phenomena. So, that would be the recommended choice for this design.



FP-6601

Figure 7.1. Continuous Stirred-Tank Reactor



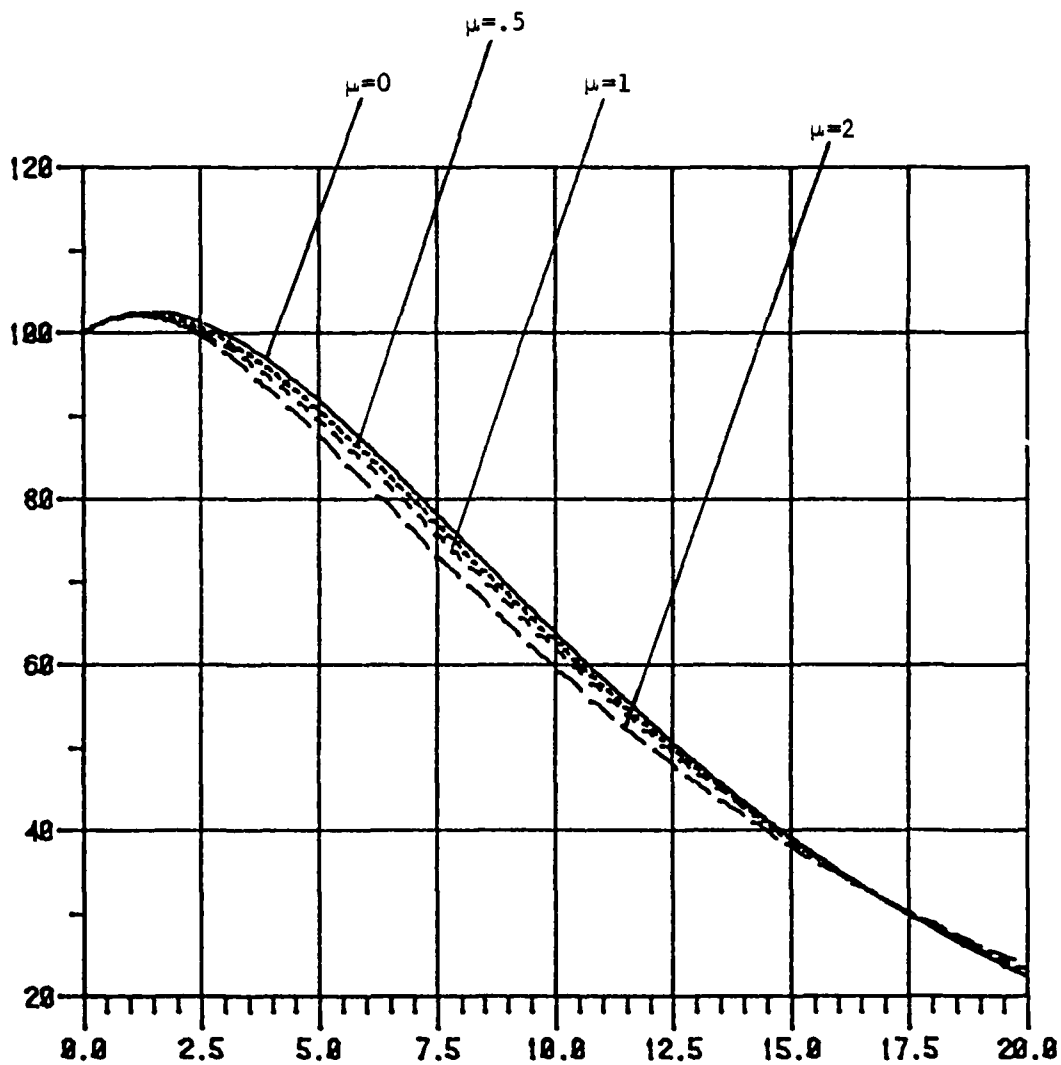


Figure 7.2.  $x_1(t)$  for R, Q and S given by equations (7.10), (7.11) and (7.12)

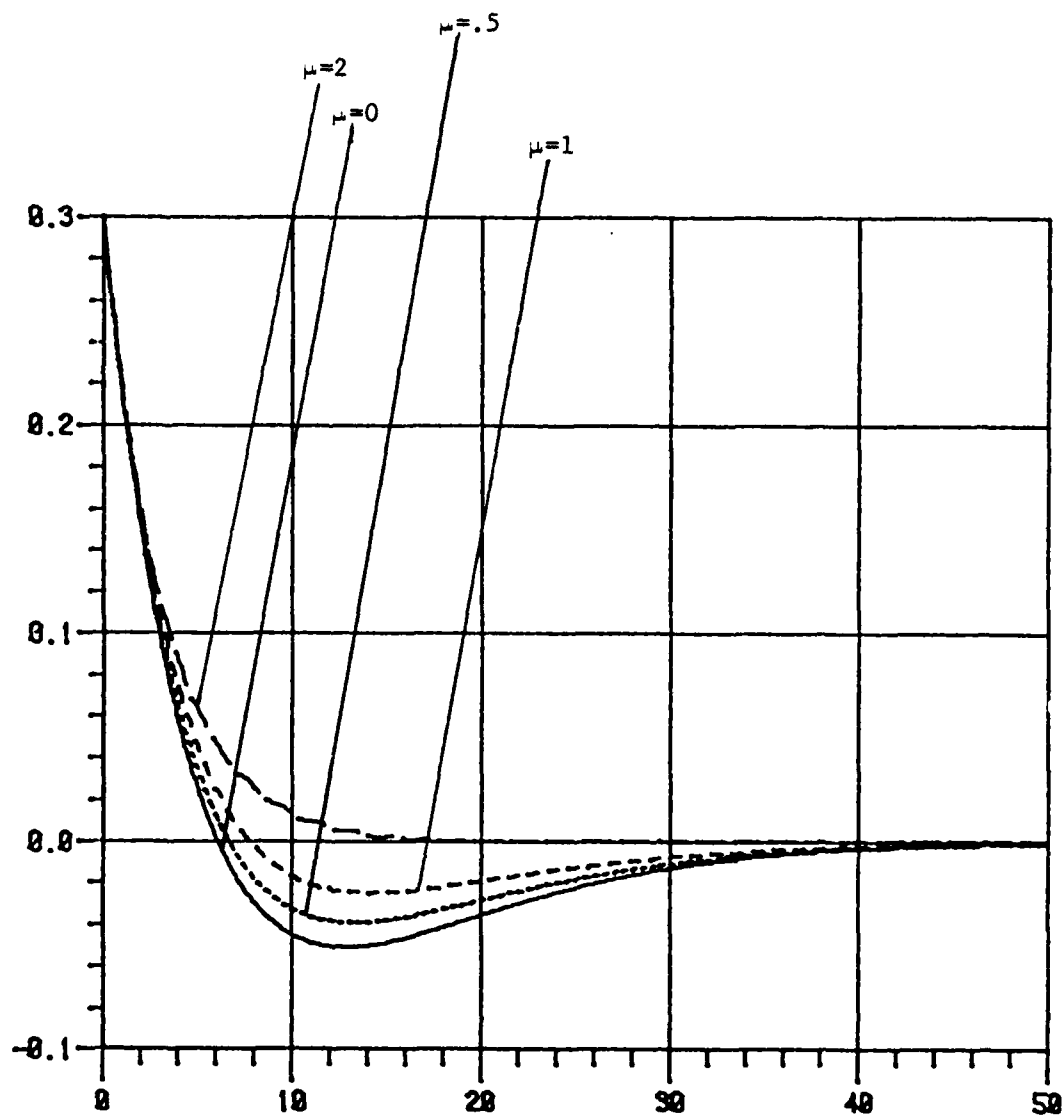


Figure 7.3.  $x_2(t)$  for R, Q and S given by equations (7.10), (7.11) and (7.12)

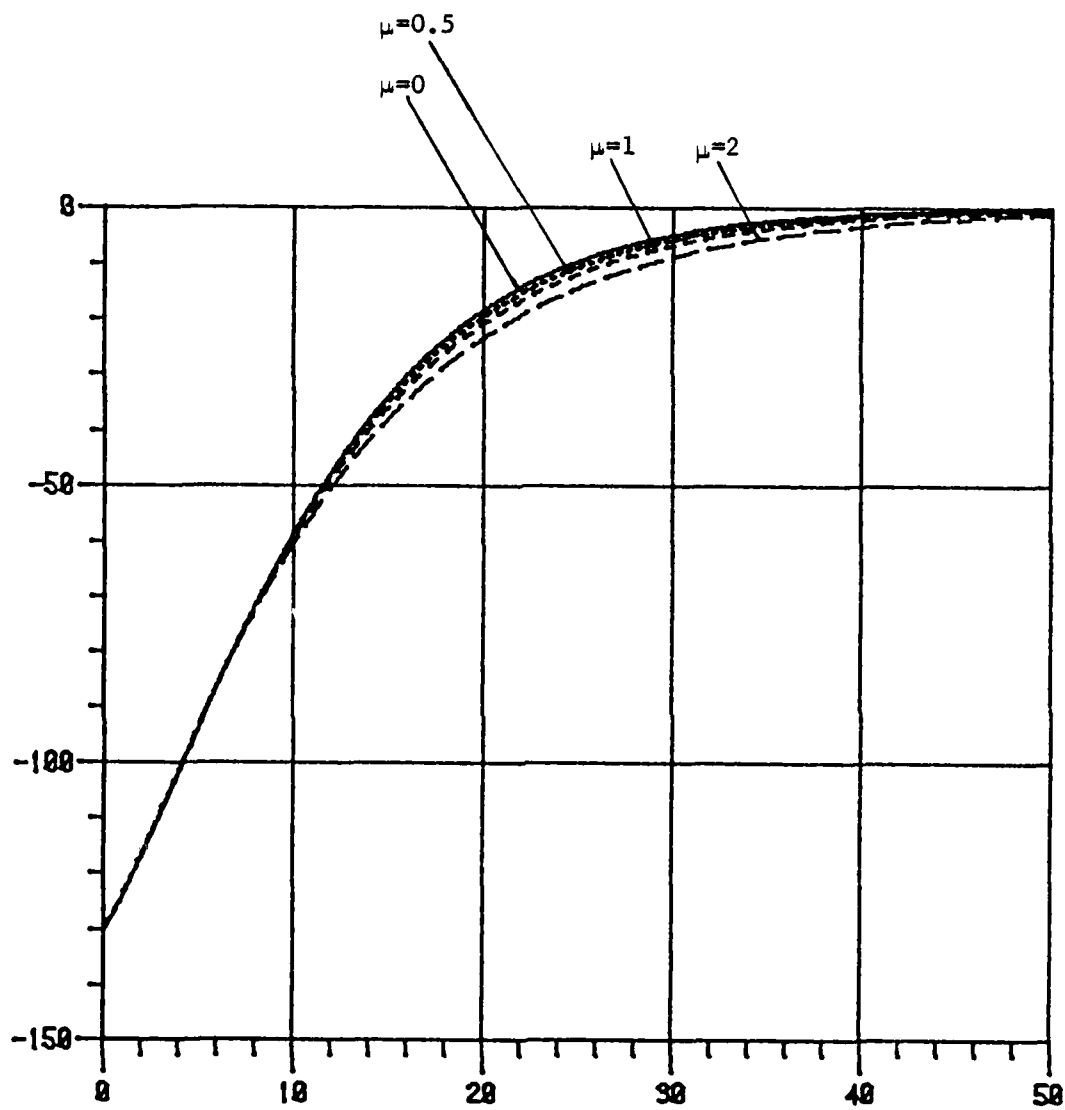


Figure 7.4.  $u_1(t)$  for  $R, Q$  and  $S$  given by equations (7.10), (7.11) and (7.12)

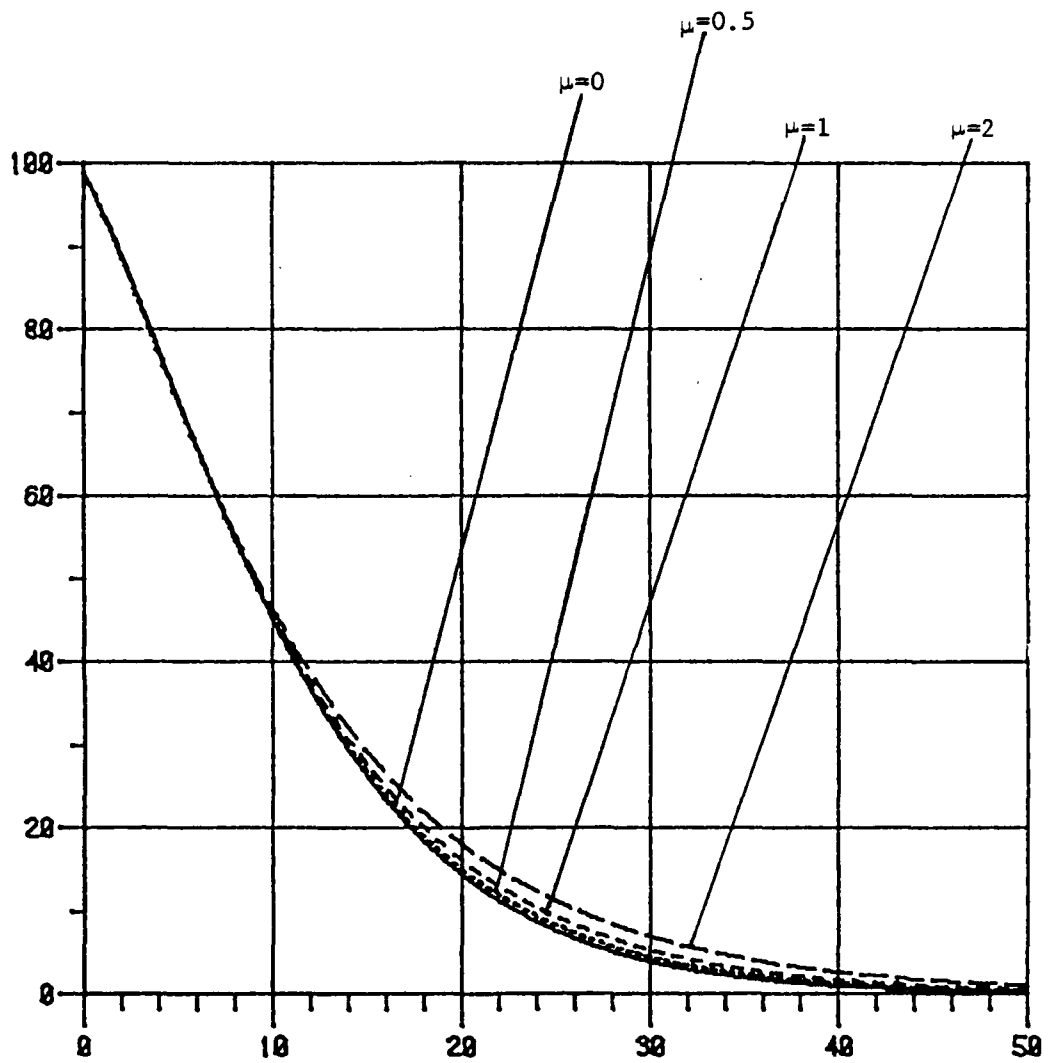


Figure 7.5.  $u_2(t)$  for R, Q and S given by equations (7.10), (7.11) and (7.12)

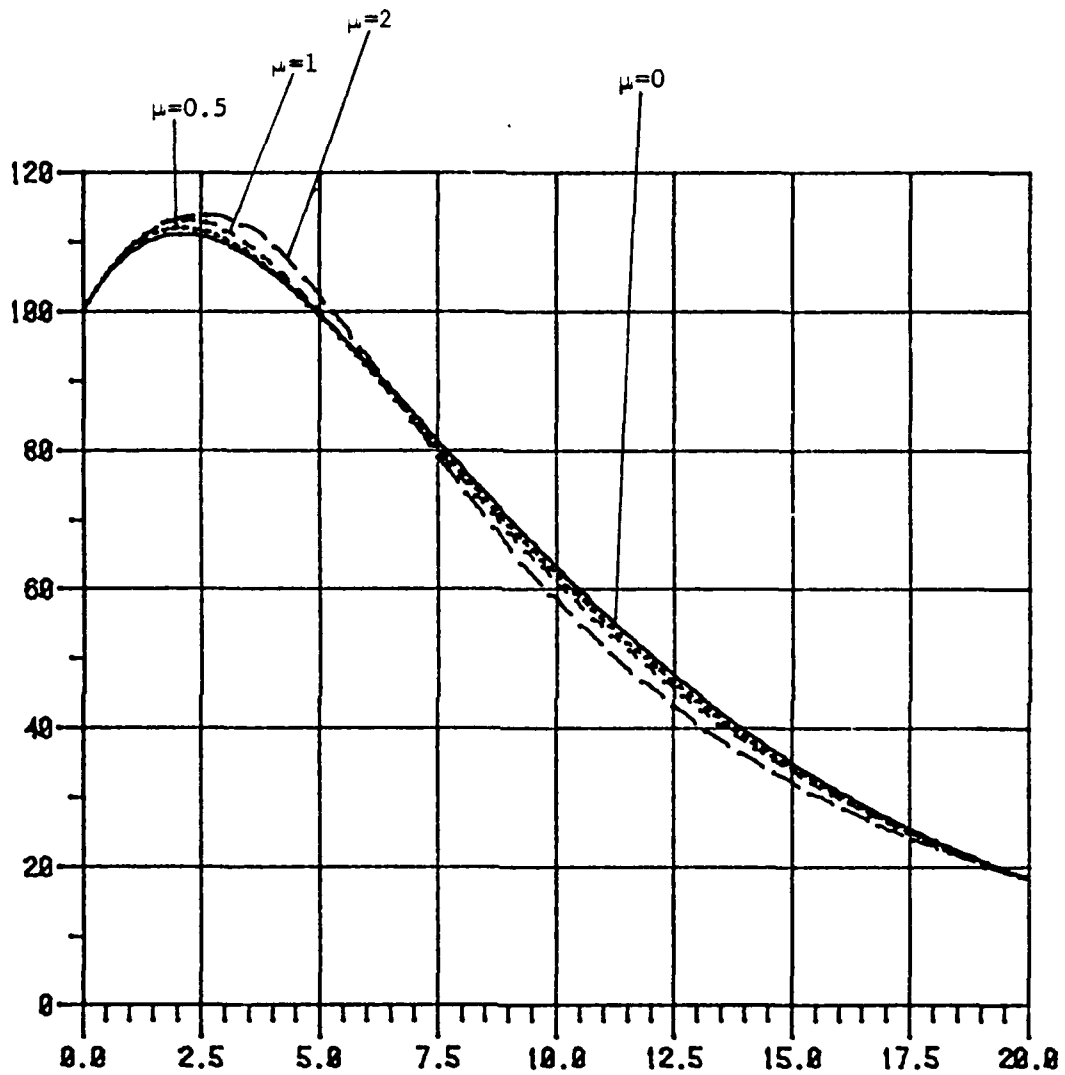


Figure 7.6.  $x_1(t)$  for R, Q and S given by equations (7.10), (7.11) and (7.14)

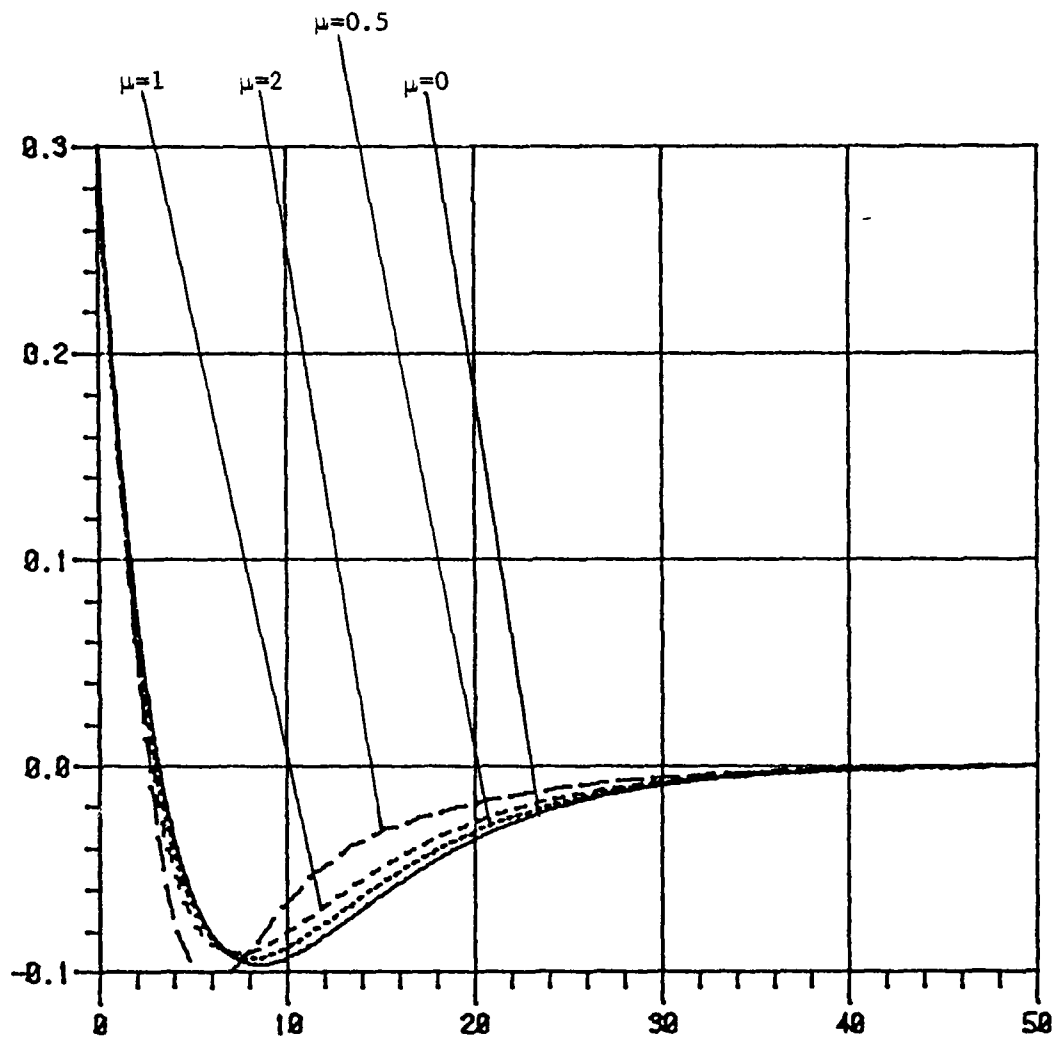


Figure 7.7.  $x_2(t)$  for R, Q and S given by equations (7.10), (7.11) and (7.14)

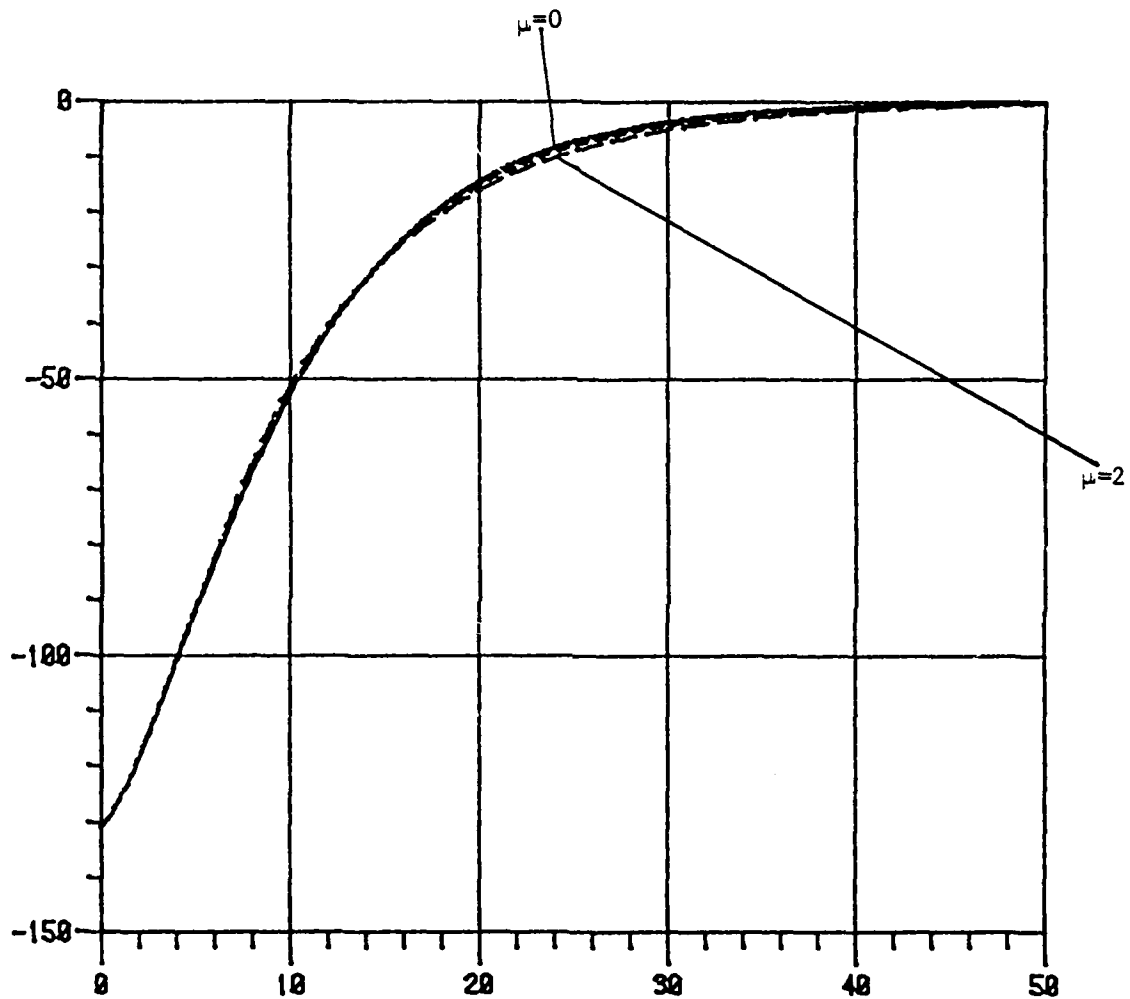


Figure 7.8.  $u_1(t)$  for R, Q and S given by equations (7.10), (7.11) and (7.14)

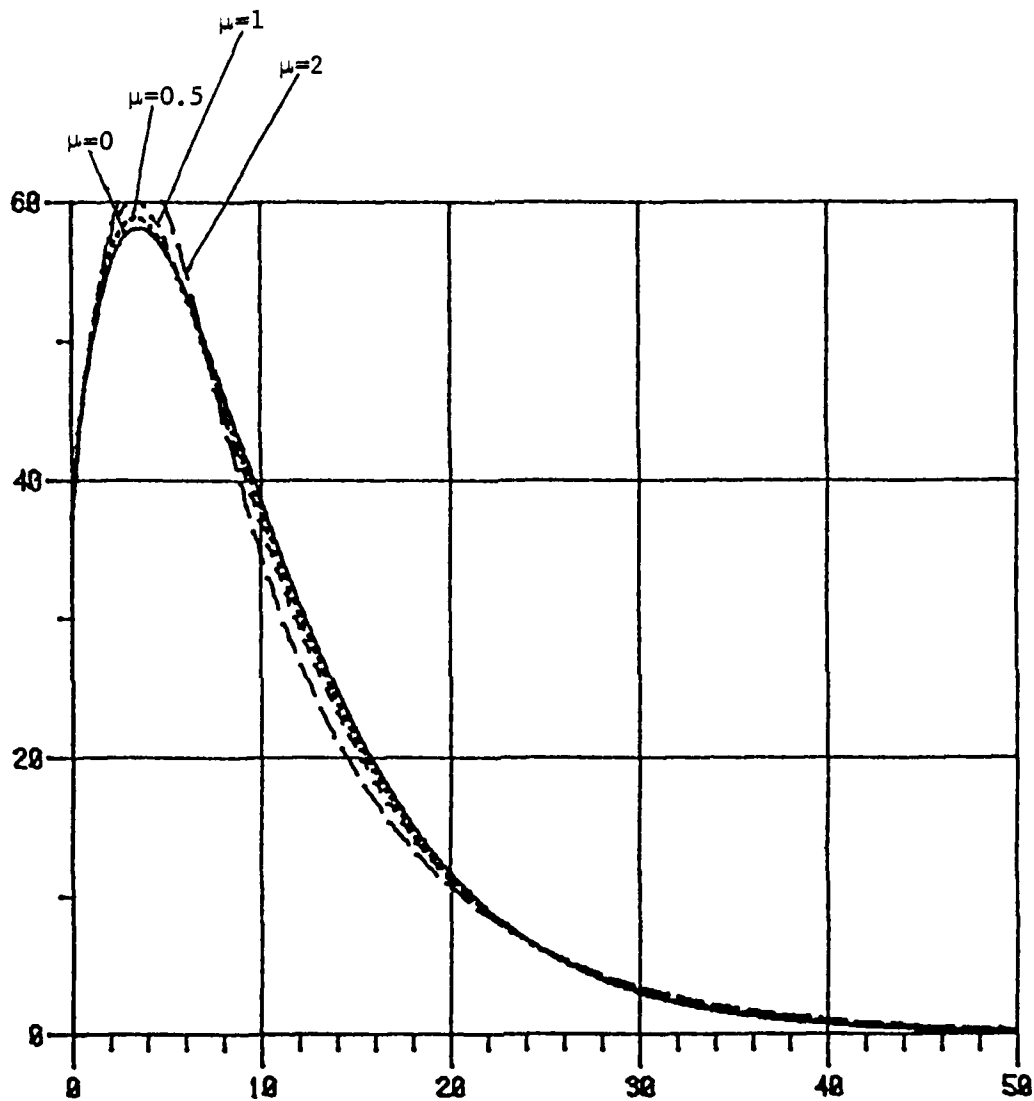


Figure 7.9.  $u_2(t)$  for R, Q and S given by equations (7.10), (7.11) and (7.14)



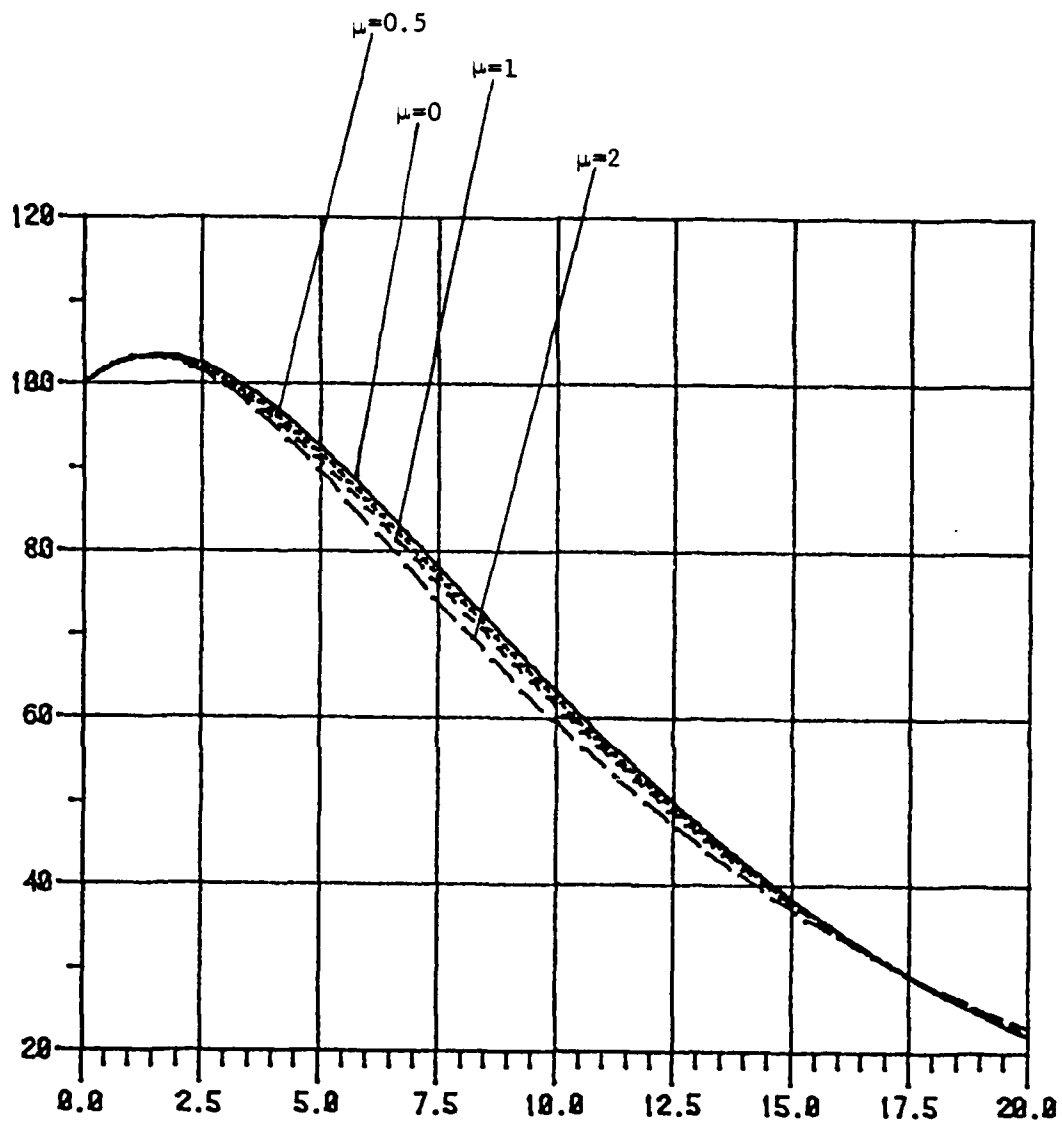


Figure 7.10.  $x_1(t)$  for  $R, Q$  and  $S$  given by equations (7.16), (7.17) and (7.18)

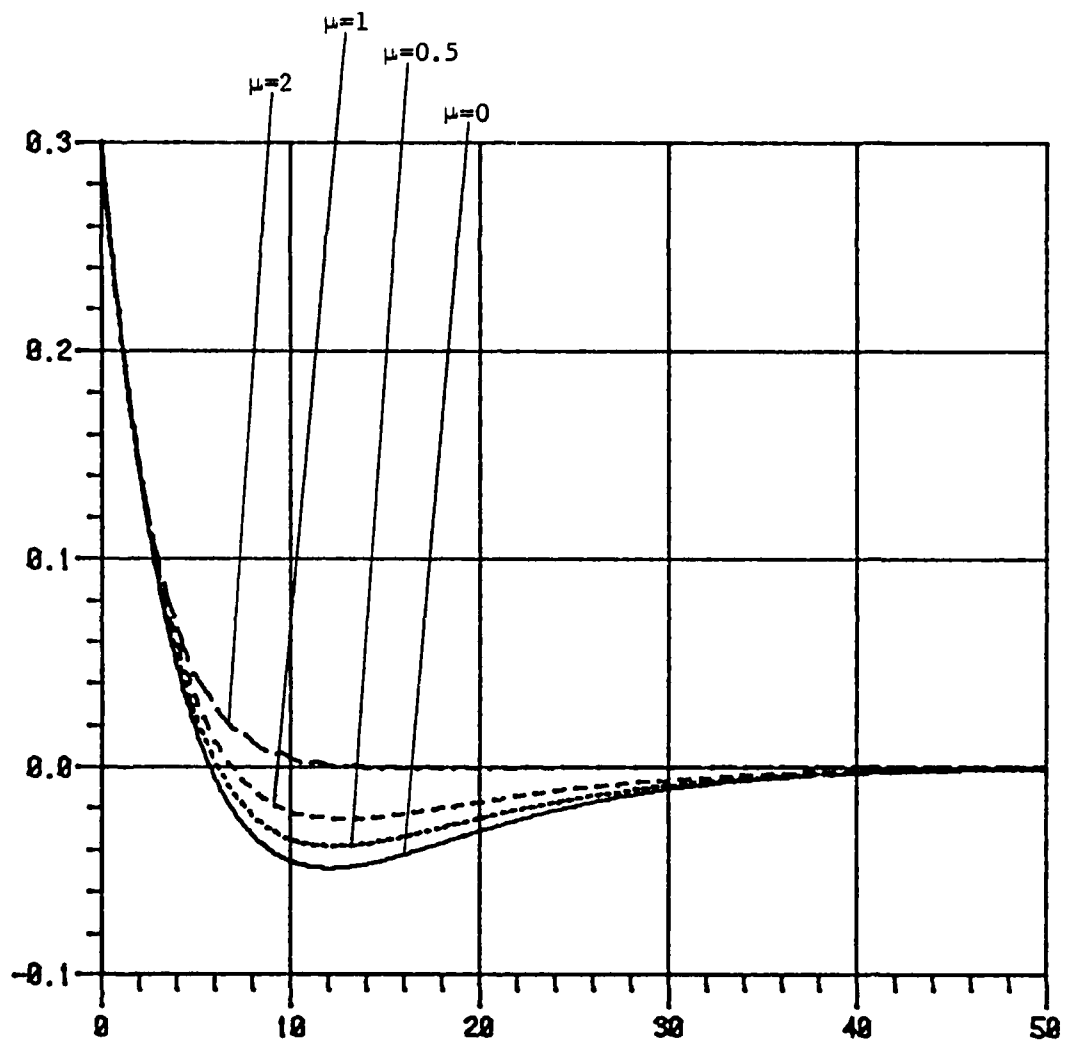


Figure 7.11.  $x_2(t)$  for R, Q and S given by equations (7.16), (7.17) and (7.18)

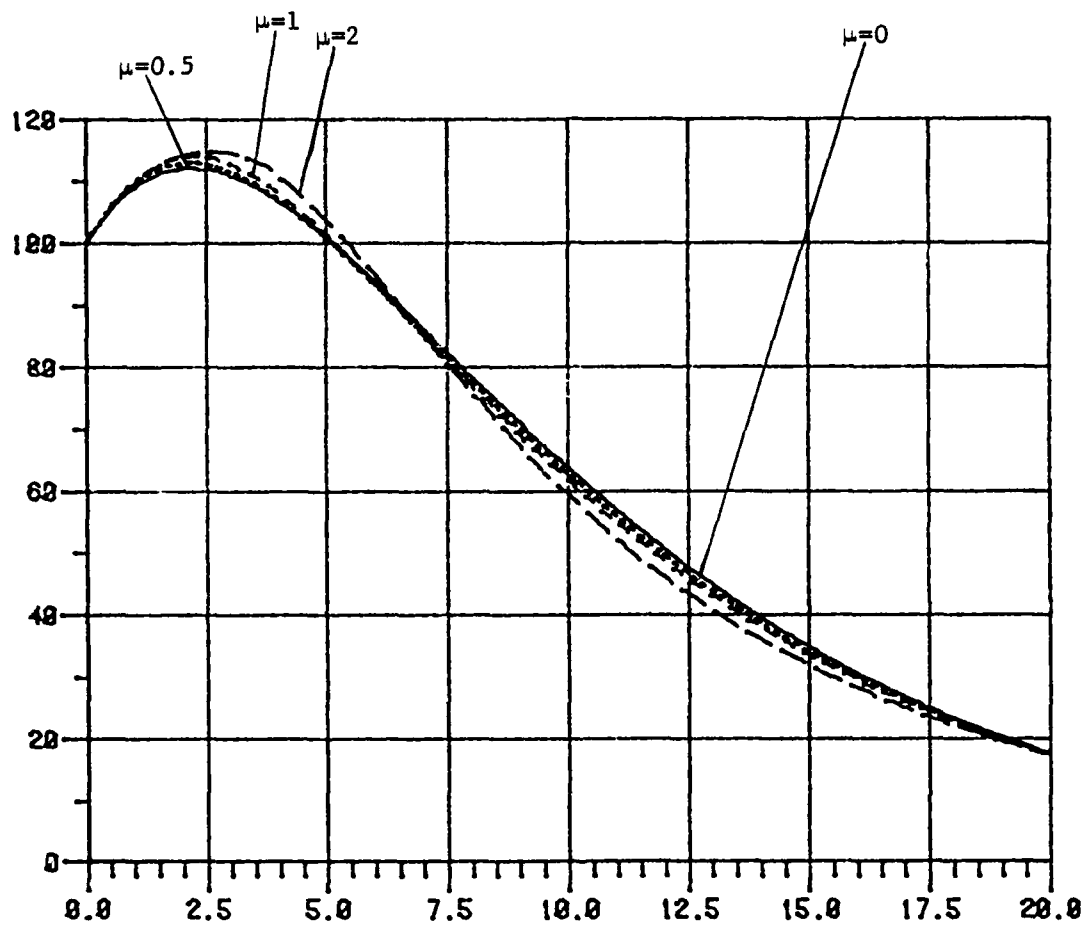


Figure 7.12.  $x_1(t)$  for R, Q and S given by equations (7.16), (7.17) and (7.20)

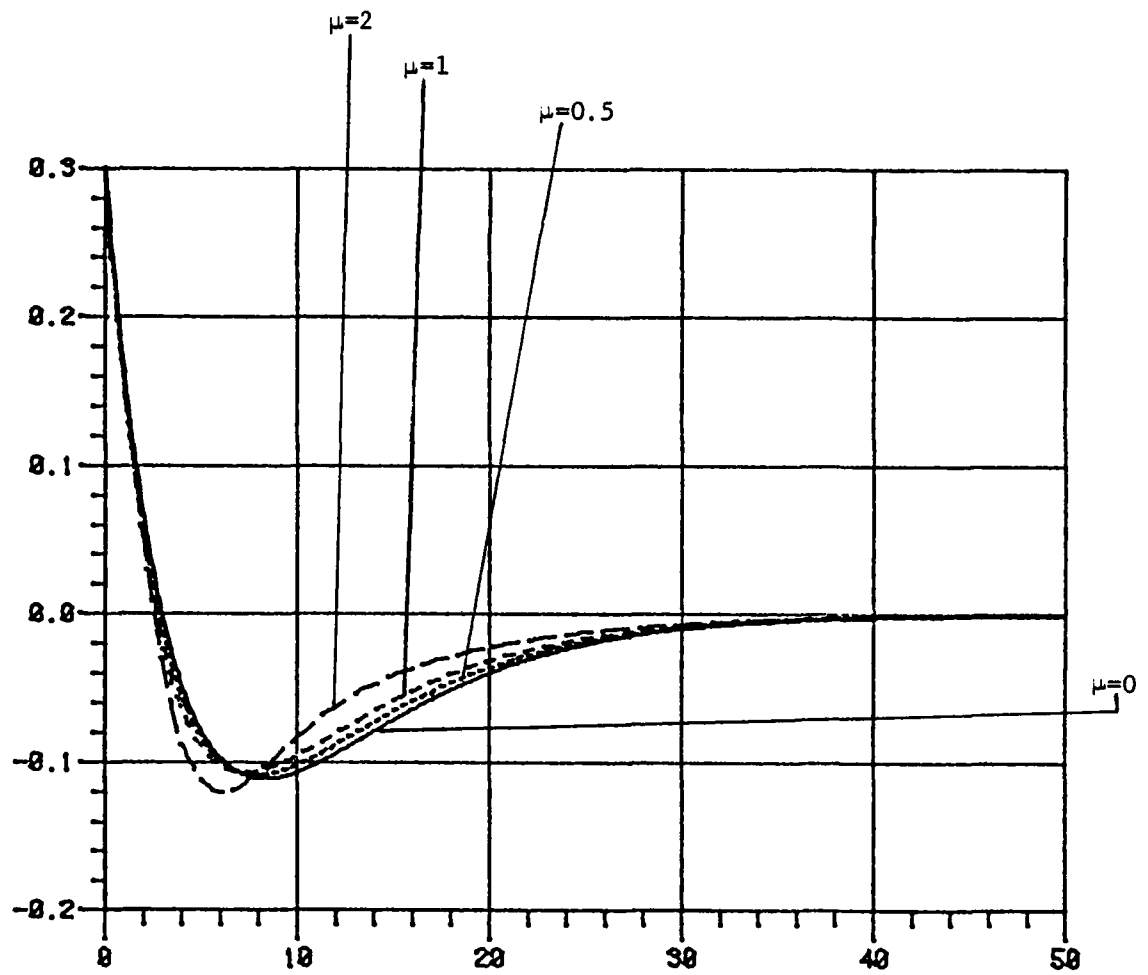


Figure 7.13.  $x_2(t)$  for R, Q and S given by equations (7.16), (7.17) and (7.20)

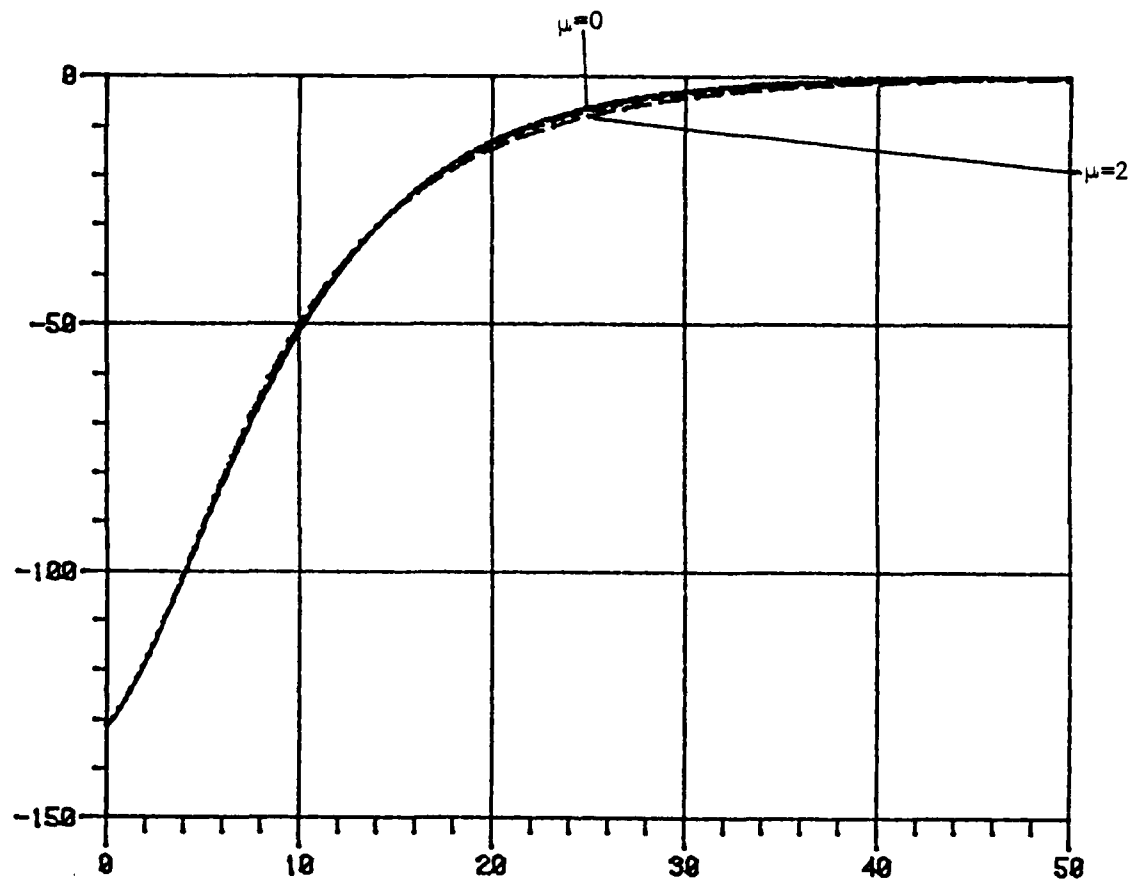


Figure 7.14.  $u_1(t)$  for  $R, Q$  and  $S$  given by equations (7.16), (7.17) and (7.20)

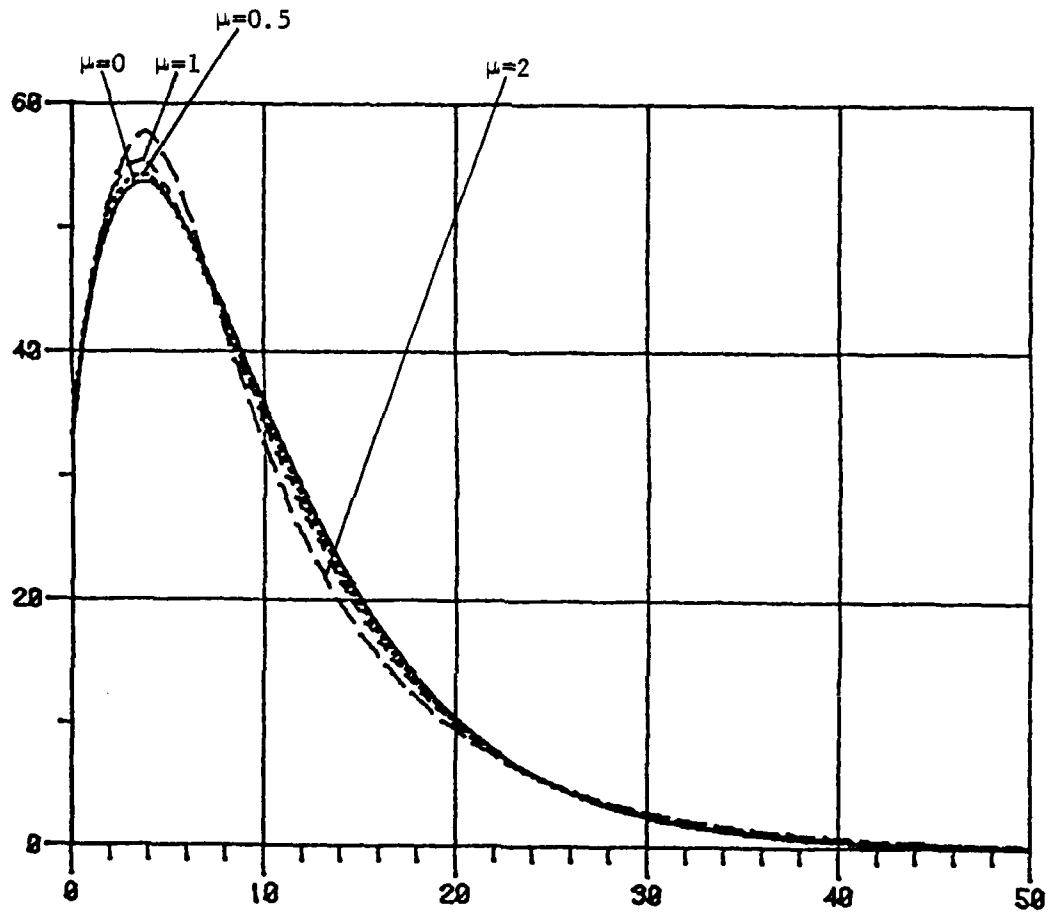


Figure 7.15.  $u_2(t)$  for R, Q and S given by equations (7.16), (7.17) and (7.20)

□

## 7.2. F100 Turbofan Engine

Consider the Pratt & Whitney F100-PW-100 afterburning turbofan, a low-bypass- ratio, twin-spool, axial-flow engine. An extensive set of linear state descriptions of this engine were given by Miller and Hackney [26]. The engine is described by a sixteenth order state model at 20 operating points. That model is for zero altitude and for a power level angle (PLA) of 67 degrees which is near maximum non-afterburning power. Such an operating point is chosen because every engine has to pass through this condition as, for example, on takeoff. The engine state variables are defined as follows

- $X_1$  = Fan Speed, SNFAN ( $N_1$ ) - rpm
- $X_2$  = Compressor Speed, SNCOM ( $N_2$ ) - rpm
- $X_3$  = Compressor Discharge Pressure,  $P_{t3}$  - psia
- $X_4$  = Interturbine Volume Pressure,  $P_{t4.5}$  - psia
- $X_5$  = Augmentor Pressure,  $P_{t7m}$  - psia
- $X_6$  = Fan Inside Diameter Discharge Temperature,  $T_{t2.5h}$  - °R
- $X_7$  = Duct Temperature,  $T_{t2.5c}$  - °R
- $X_8$  = Compressor Discharge Temperature,  $T_{t3}$  - °R
- $X_9$  = Burner Exit Fast Response Temperature,  $T_{t4hi}$  - °R
- $X_{10}$  = Burner Exit Slow Response Temperature,  $T_{t4lo}$  - °R
- $X_{11}$  = Burner Exit Total Temperature,  $T_{t4}$  - °R
- $X_{12}$  = Fan Turbine Inlet Fast Response Temperature,  $T_{t4.5hi}$  - °R
- $X_{13}$  = Fan Turbine Inlet Slow Response Temperature,  $T_{t4.5lo}$  - °R
- $X_{14}$  = Fan Turbine Exit Temperature,  $T_{t5}$  - °R
- $X_{15}$  = Duct Exit Temperature,  $T_{t6c}$  - °R
- $X_{16}$  = Duct Exit Temperature,  $T_{t7m}$  - °R

The inputs are defined as follows

- $U_1$  = Main Burner Fuel Flow, WFMB - lb/hr
- $U_2$  = Nozzle Jet Area,  $A_j$  - ft<sup>2</sup>
- $U_3$  = Inlet Guide Vane Position, CIVV - deg
- $U_4$  = High Variable Stator Position, RCVV - deg
- $U_5$  = Customer Compressor Bleed Flow, BLC - %

The outputs are defined as follows

- $Y_1$  = Engine Net Thrust Level, FN - lb
- $Y_2$  = Total Engine Airflow, WFAN - lb/sec
- $Y_3$  = Turbine Inlet Temperature,  $T_{t4}$  - °R
- $Y_4$  = Fan Stall Margin, SMAF
- $Y_5$  = Compressor Stall Margin, SMHC

The sixteen eigenvalues of the engine were determined to be approximately: -577, -175, -58, -51, -48, -39,  $-21.4 \pm j0.9$ , -18.6,  $-17.8 \pm j4.2$ ,  $-5.8 \pm j5$ , -3.8, -2, -.68.

A reduced order plant was obtained [26] by eliminating all eigenvalues beyond the frequency range of interest, namely all real parts less than -17.8. This gave a fifth order model. However since the fan turbine inlet temperature (FTIT) is a sensed variable, the fast response eigenvalue of FTIT (-51) was also included. A sixth order reduced engine model was established with eigenvalues and state variable correspondence as follows

- 0.68             $-T_{410}$ , Burner Exit Slow Response ( $x_6$ )
- 2.0             $-T_{4.510}$ , Fan Turbine Inlet Slow Response ( $x_5$ )
- 4.06             $-N_2$ , Compressor Speed ( $x_2$ )



-5.4 + 4.71  $N_1$ , Fan Speed ( $x_1$ )

-5.4 - 4.71  $P_7$ , Augmentor Pressure ( $x_3$ )

-51  $T_{4.5hi}$ , Fan Turbine Inlet Fast Response ( $x_4$ )

All of the sensed variables, except compressor discharge pressure, appear in the reduced model. Four inputs and four outputs are considered in that model. The inputs are defined as follows

$u_1$  = WF, Fuel Flow, PPH

$u_2$  = AJ, exhaust nozzle area,  $FT^2$

$u_3$  = CIVV, inlet vane position, DEG

$u_4$  = RCVV, compressor vane position, DEG

The outputs are defined as follows

$y_1$  =  $N_1$ , fan speed, RPM

$y_2$  =  $N_2$ , compressor speed, RPM

$y_3$  =  $P_7$ , augmentor pressure, PSI

$y_4$  = FTIT, fan turbine inlet temperature, °F.

The parameter matrices A, C and E are given [27] as follows

$$A = \begin{bmatrix} -4.064 & 3.895 & -470.5 & 7.971 & 5.294 & -3.005 \\ .03718 & -2.958 & -59.13 & .1727 & 2.08 & 12.48 \\ .03389 & .0067 & -4.442 & .0059 & .1474 & .0985 \\ 1.164 & -2.646 & -331.6 & -50.05 & -.473 & -11.36 \\ .05174 & -.1176 & -14.74 & -2.001 & -2.021 & -.505 \\ .00184 & .0036 & -.601 & .00008 & .0009 & -.666 \end{bmatrix} \quad (7.22)$$

$$C = \begin{bmatrix} .8686 & -14.51 & -96.14 & 9.246 \\ .9096 & -58.46 & -1.053 & -60.15 \\ -.007994 & -79.66 & 1.2 & .3673 \\ 5.643 & -112.2 & -18.23 & 41.53 \\ .2508 & -4.99 & -.8106 & 1.846 \\ .01 & -.3166 & -.02915 & .07426 \end{bmatrix} \quad (7.23)$$

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \quad (7.24)$$

Consider the problem of controlling the above reduced sixth order model, using an output feedback control law so as to minimize a quadratic performance index. To implement such a feedback control a time delay may be expected due to the measurement of the output variables. Preliminary analysis of this example has shown overshoots in the trajectories of  $x_1$  and  $x_2$  at  $t = 0.005$ . So, we expect a time delay of this order of magnitude to have a significant effect on the system trajectories. It has been seen that delays of the order of  $10^{-4}$  result in a significant deviation of the trajectories from the nominal ones. Larger delays result in completely destabilizing the system. To reduce that effect we apply the design strategy described in Section 4.2.

Consider the following parameters

$$R = 10^4 I_4 \quad (7.25)$$

$$Q = I_6 \quad (7.26)$$

$$S = 0 \quad (7.27)$$

The feedback gain matrix is

$$D = \begin{bmatrix} -.82 \times 10^{-6} & -.26 \times 10^{-5} & .58 \times 10^{-4} & .76 \times 10^{-7} \\ -.15 \times 10^{-2} & -.61 \times 10^{-3} & .83 \times 10^{-1} & .63 \times 10^{-4} \\ .74 \times 10^{-4} & .21 \times 10^{-5} & -.3 \times 10^{-2} & .14 \times 10^{-4} \\ -.10 \times 10^{-4} & .17 \times 10^{-3} & -.11 \times 10^{-2} & -.21 \times 10^{-4} \end{bmatrix} \quad (7.28)$$

Two values of  $\mu$  are simulated in addition to the nominal value, i.e. zero. The state trajectories are illustrated in Figures 7.16 - 7.21. It is seen that the first three state trajectories show high sensitivity. So, we choose  $S$  with suitable entries so as to penalize their sensitivities more than the other three.

Now, consider the same values of  $R$  and  $Q$  given by equations (7.25) and (7.26) along with the following value of  $S$ :

$$S = \text{diag}[100, 100, 100, 10, 10, 10] \quad (7.29)$$

The feedback gain matrix in this case is

$$D = \begin{bmatrix} -.76 \times 10^{-6} & -.21 \times 10^{-5} & .38 \times 10^{-4} & .69 \times 10^{-7} \\ -.21 \times 10^{-2} & -.68 \times 10^{-3} & .87 \times 10^{-1} & .57 \times 10^{-4} \\ .54 \times 10^{-4} & .11 \times 10^{-5} & -.36 \times 10^{-2} & .11 \times 10^{-4} \\ -.20 \times 10^{-4} & .14 \times 10^{-3} & -.10 \times 10^{-2} & -.19 \times 10^{-4} \end{bmatrix} \quad (7.30)$$

The state trajectories illustrated in Figures 7.22 - 7.27. The nominal trajectories of this case are illustrated on a wider horizon in Figures 7.28 - 7.33 to demonstrate the stability of the system.

In order to obtain more satisfactory sensitivity consider the same values of  $R$  and  $Q$  given by equations (7.25) and (7.26) along with the follow-

ing values of  $S$

$$S = [1000, 1000, 1000, 100, 100, 100] \quad (7.31)$$

The feedback gain matrix in this case is

$$D = \begin{bmatrix} -.73 \times 10^{-6} & -.11 \times 10^{-5} & .21 \times 10^{-4} & .60 \times 10^{-7} \\ -.23 \times 10^{-2} & -.68 \times 10^{-3} & .88 \times 10^{-1} & .51 \times 10^{-4} \\ .49 \times 10^{-4} & .11 \times 10^{-5} & -.38 \times 10^{-2} & .11 \times 10^{-4} \\ -.23 \times 10^{-4} & .13 \times 10^{-3} & -.10 \times 10^{-2} & -.10 \times 10^{-4} \end{bmatrix} \quad (7.32)$$

The state trajectories are illustrated in Figures 7.34 - 7.39.

From the plots associated with the above cases, we see that sensitivity to small time delays is reduced due to our proposed design strategy. We notice a significant improvement in sensitivity of the first 3 state trajectories that were penalized the most. It is observed in this example that increasing  $S$  is reflected directly on the sensitivity reduction of the different state trajectories. The choice of  $S$  given by equation (7.31) has shown satisfactory sensitivity for all of the six states as well as an acceptable transient phenomena. The choice of  $Q$  is kept unchanged in each trial of the values of  $S$  because the resulting transient phenomena was acceptable. In fact it is improved for some trajectories. For example,  $x_2(t)$  illustrated in Figure 7.35 shows an amount of undershoot which is less than that shown in Figure 7.23 which, in turn is less than the one shown in Figure 7.17. The feedback gain matrices given by equations (7.28), (7.30) and (7.32) have norms which are approximately equal which means that not much increase in the control effort is needed. So, the choice of  $R$  is unchanged with the different trials of  $S$ .

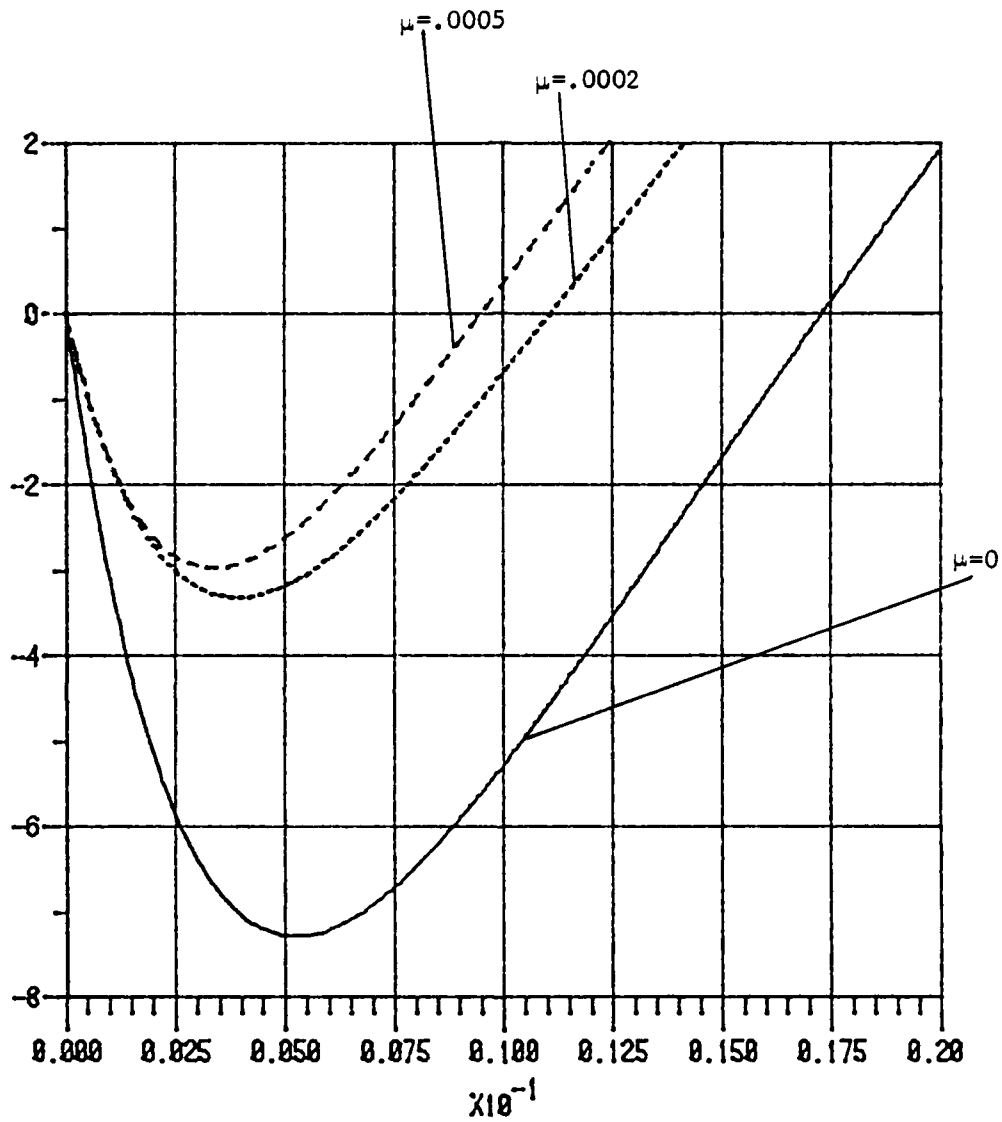


Figure 7.16.  $x_1(t)$  for  $S=0$

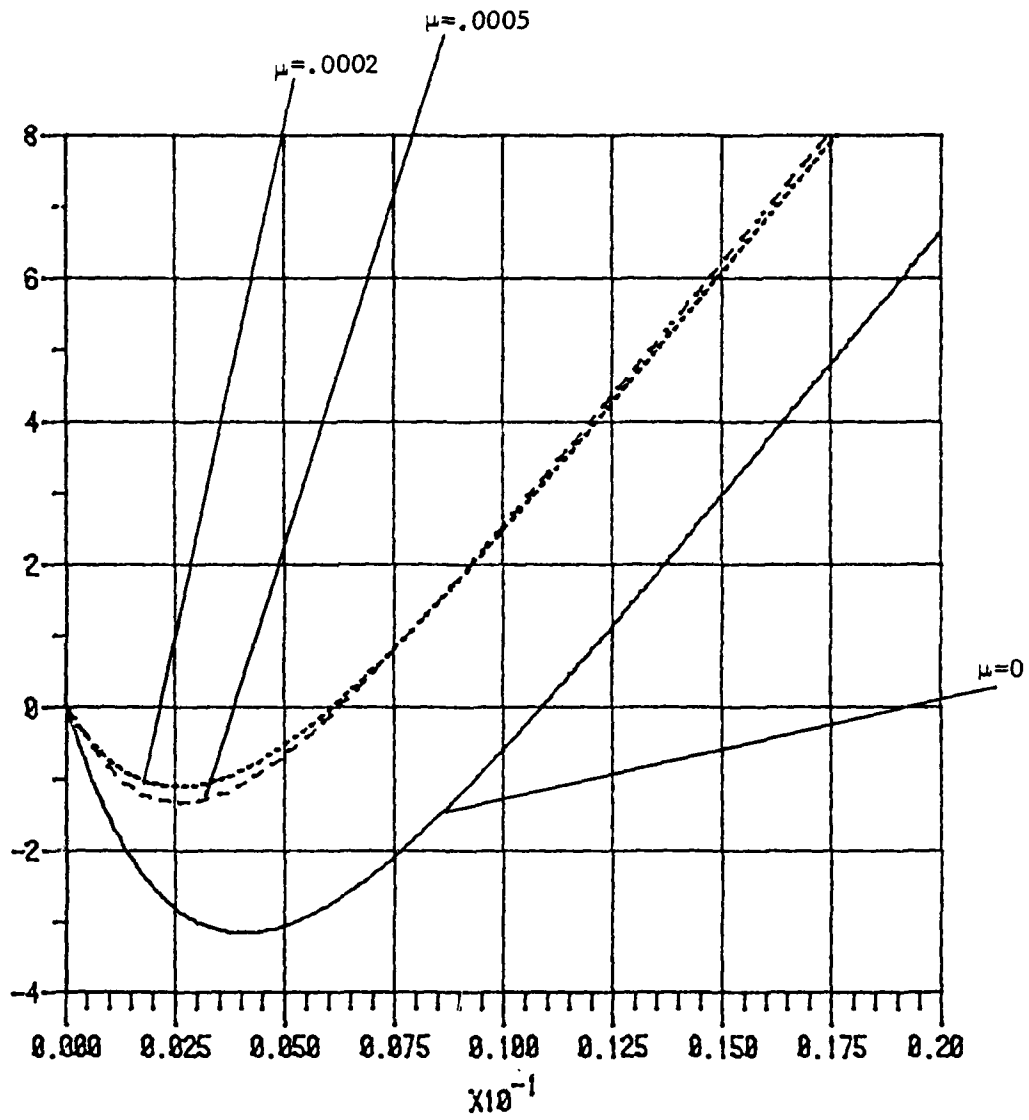


Figure 7.17.  $x_2(t)$  for  $S = 0$

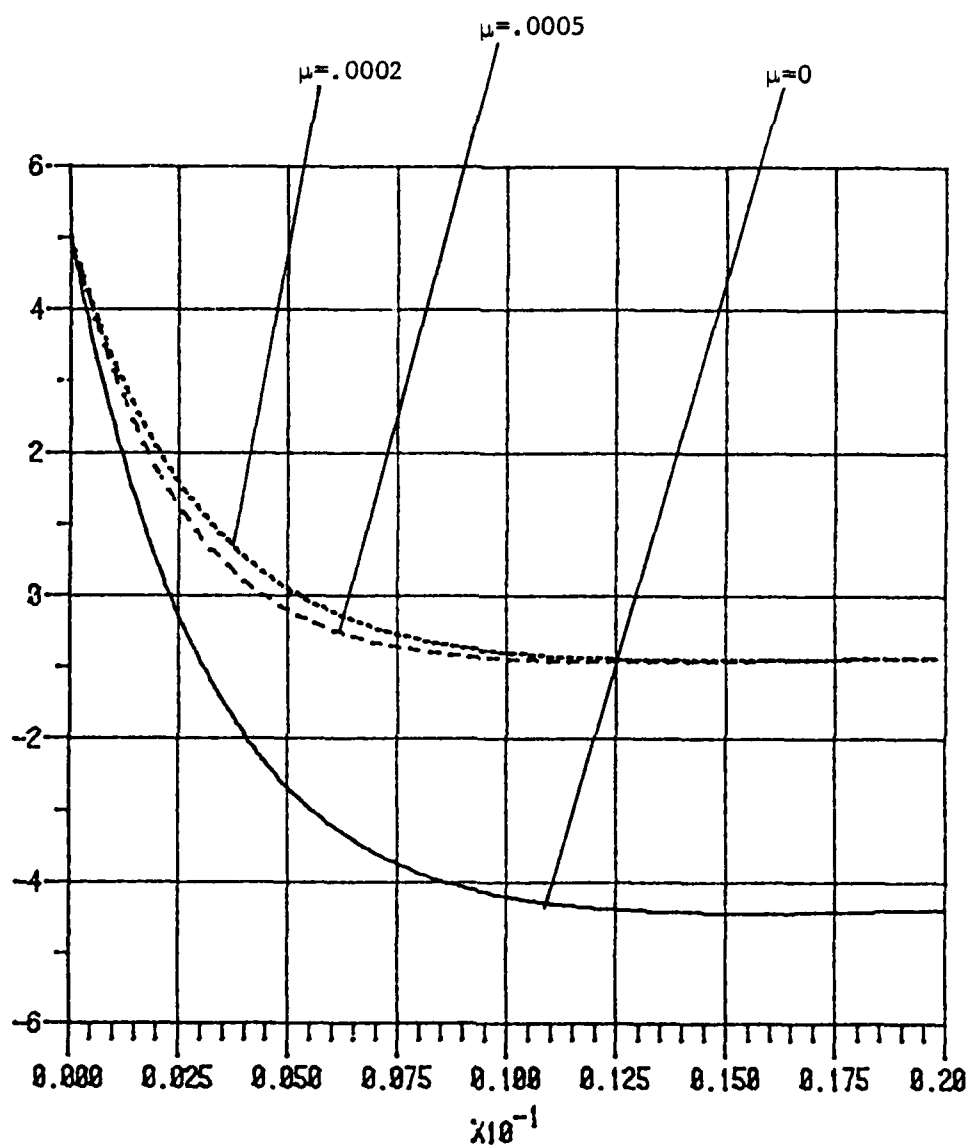


Figure 7.18.  $x_3(t)$  for  $S=0$

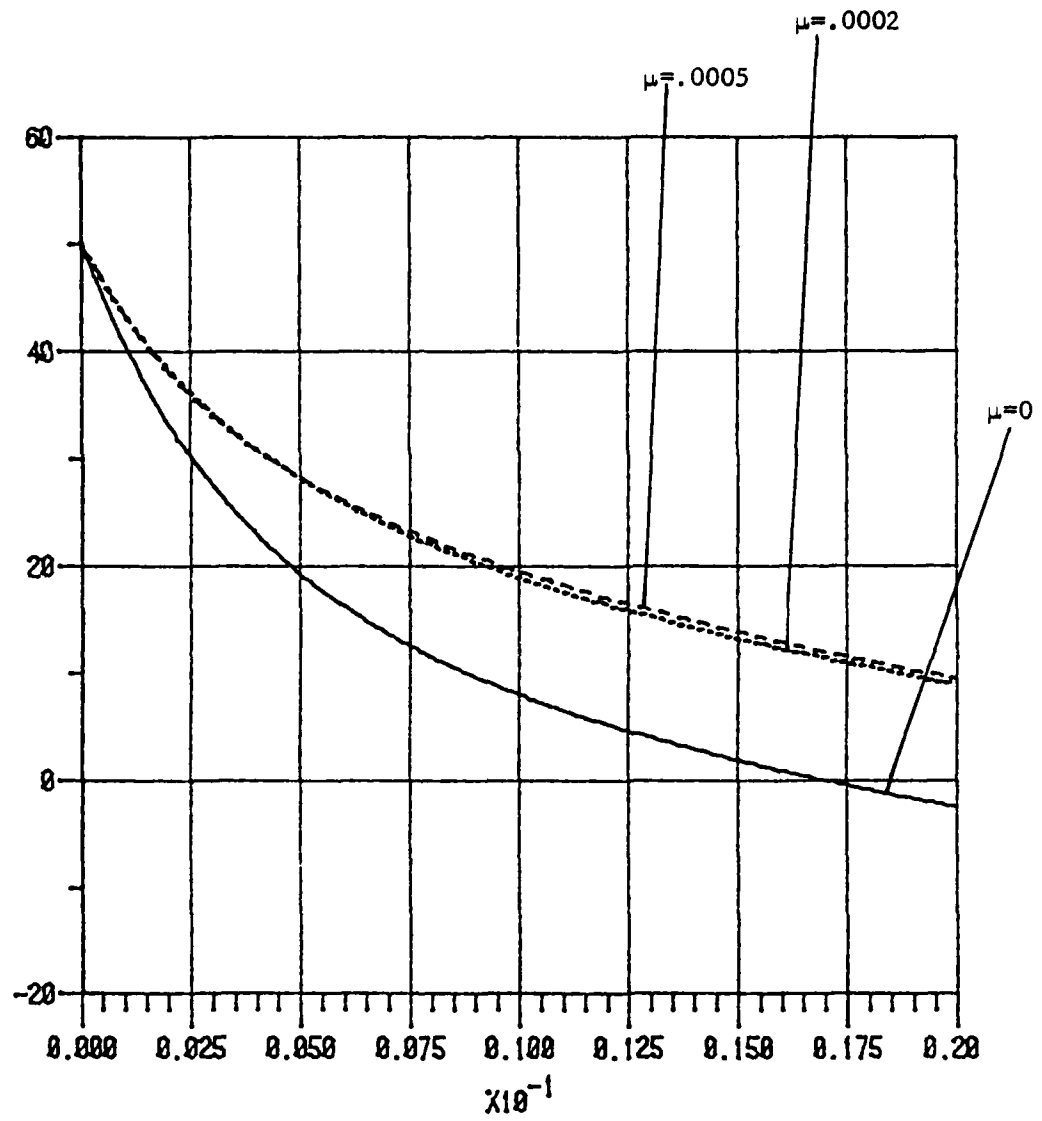


Figure 7.19.  $x_4(t)$  for  $S = 0$



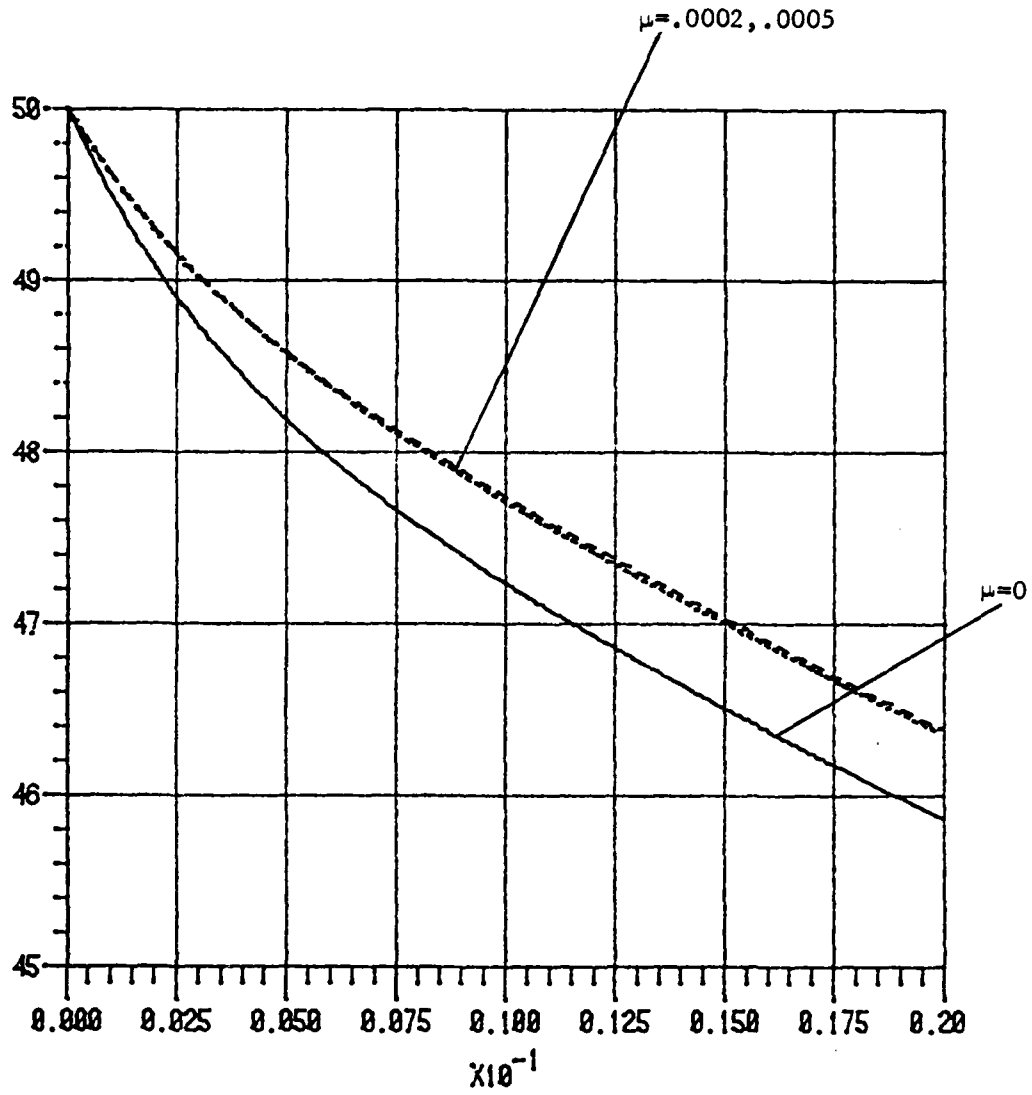


Figure 7.20.  $x_5(t)$  for  $S=0$

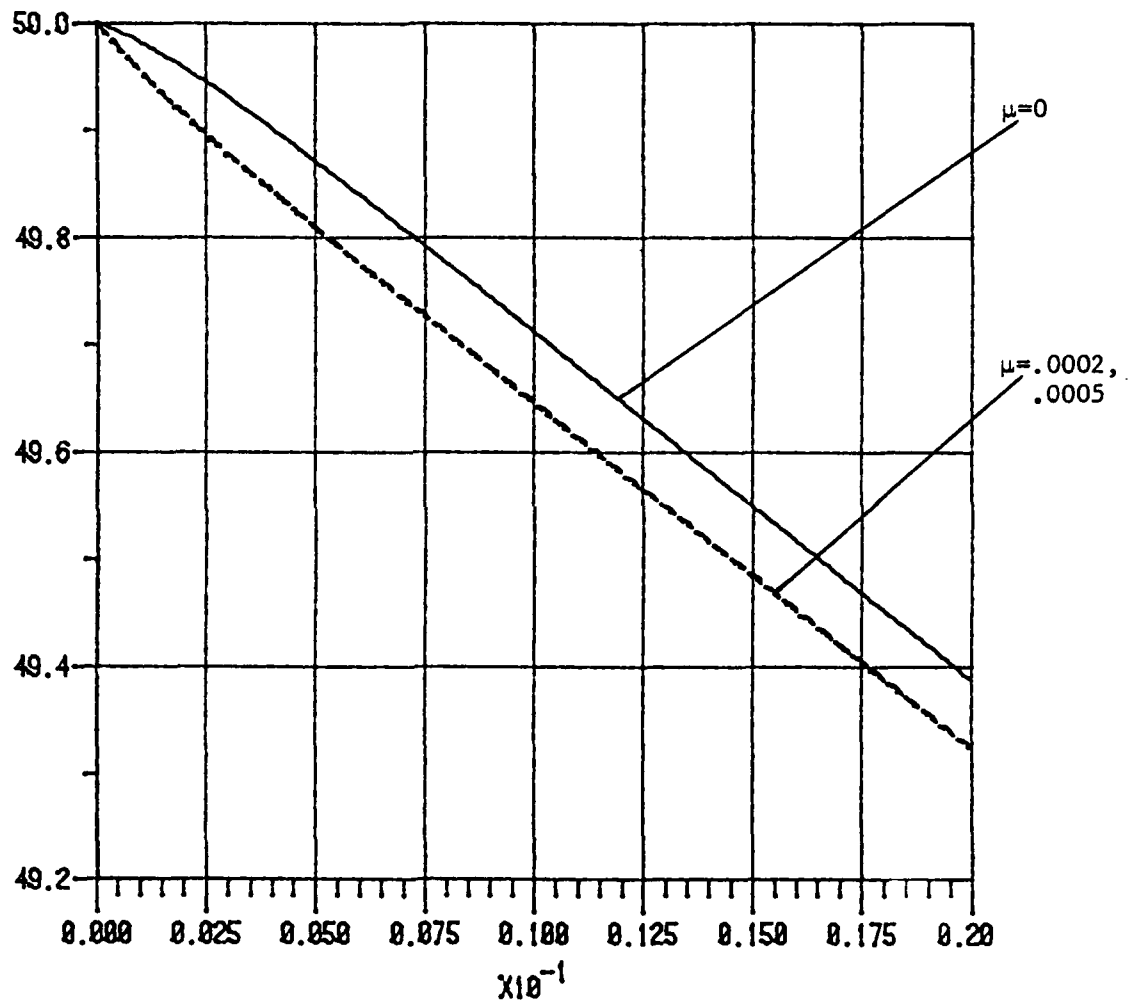


Figure 7.21.  $x_6(t)$  for  $S=0$

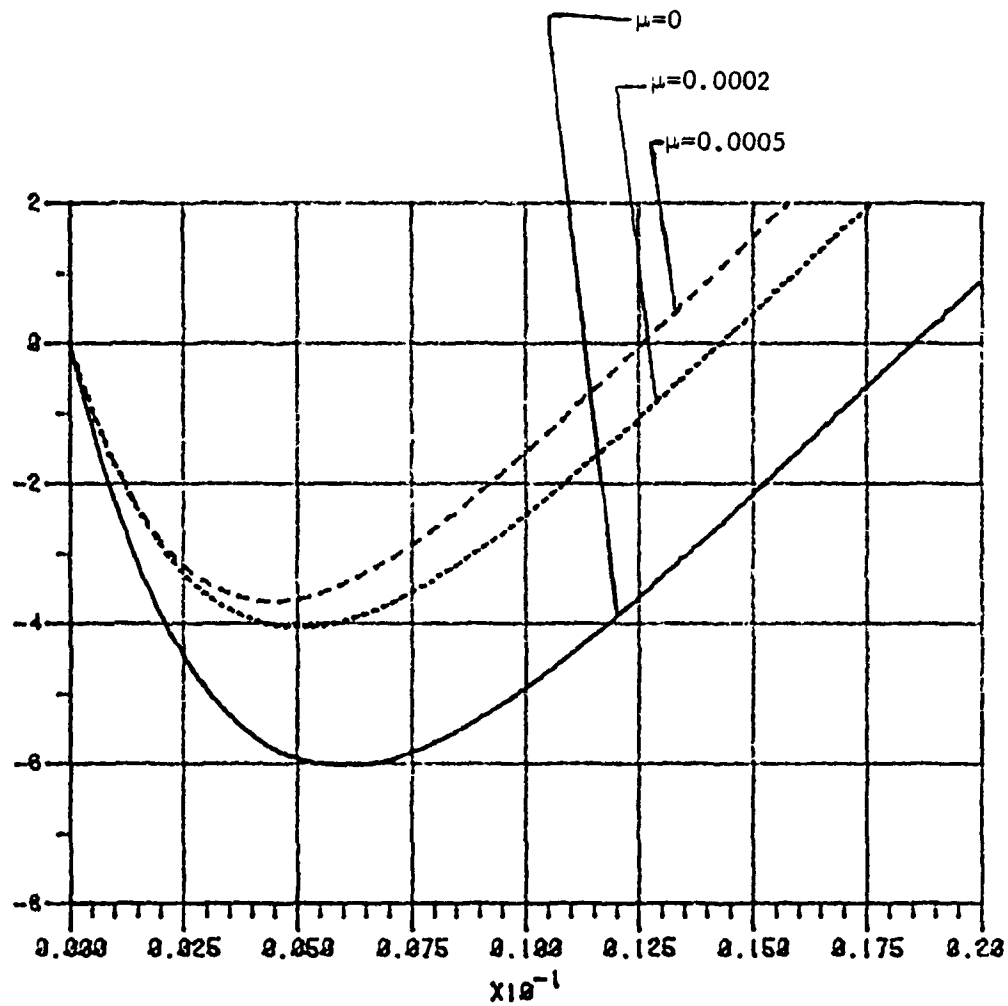


Figure 7.22.  $x_1(t)$  for  $S$  given by equation (7.29)

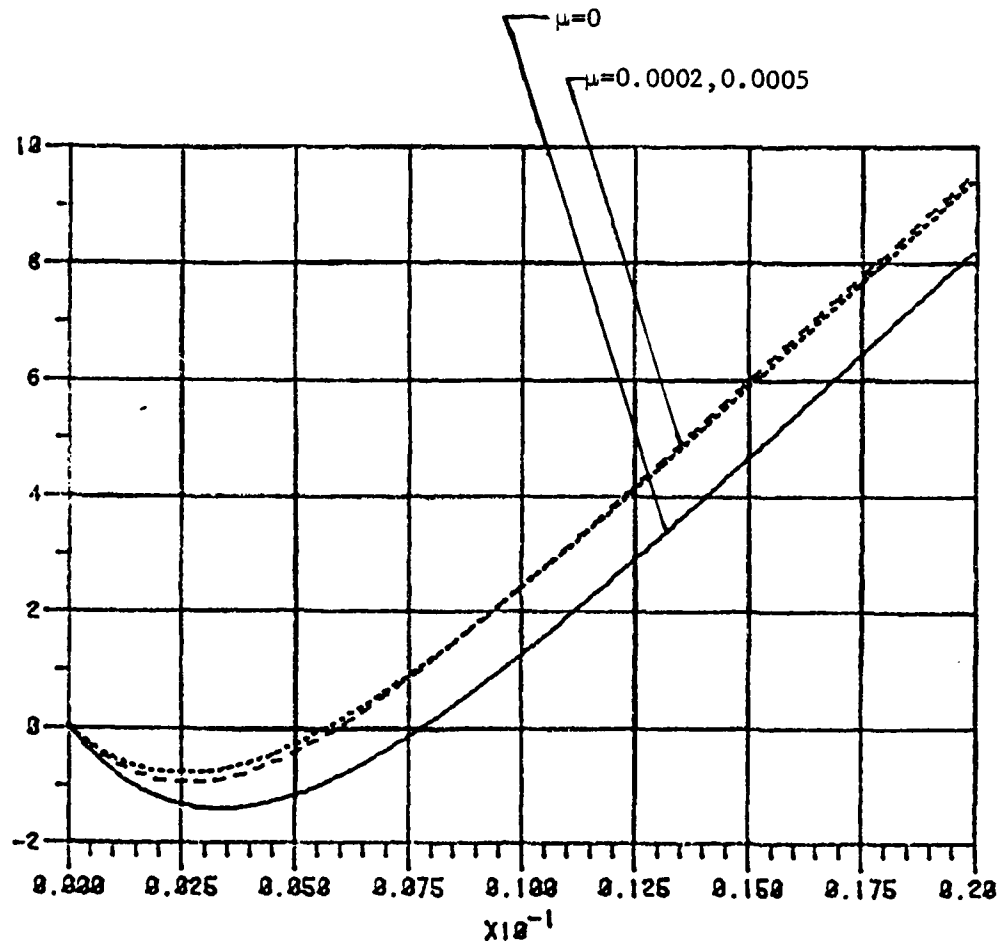


Figure 7.23.  $x_2(t)$  for  $S$  given by equation (7.29)

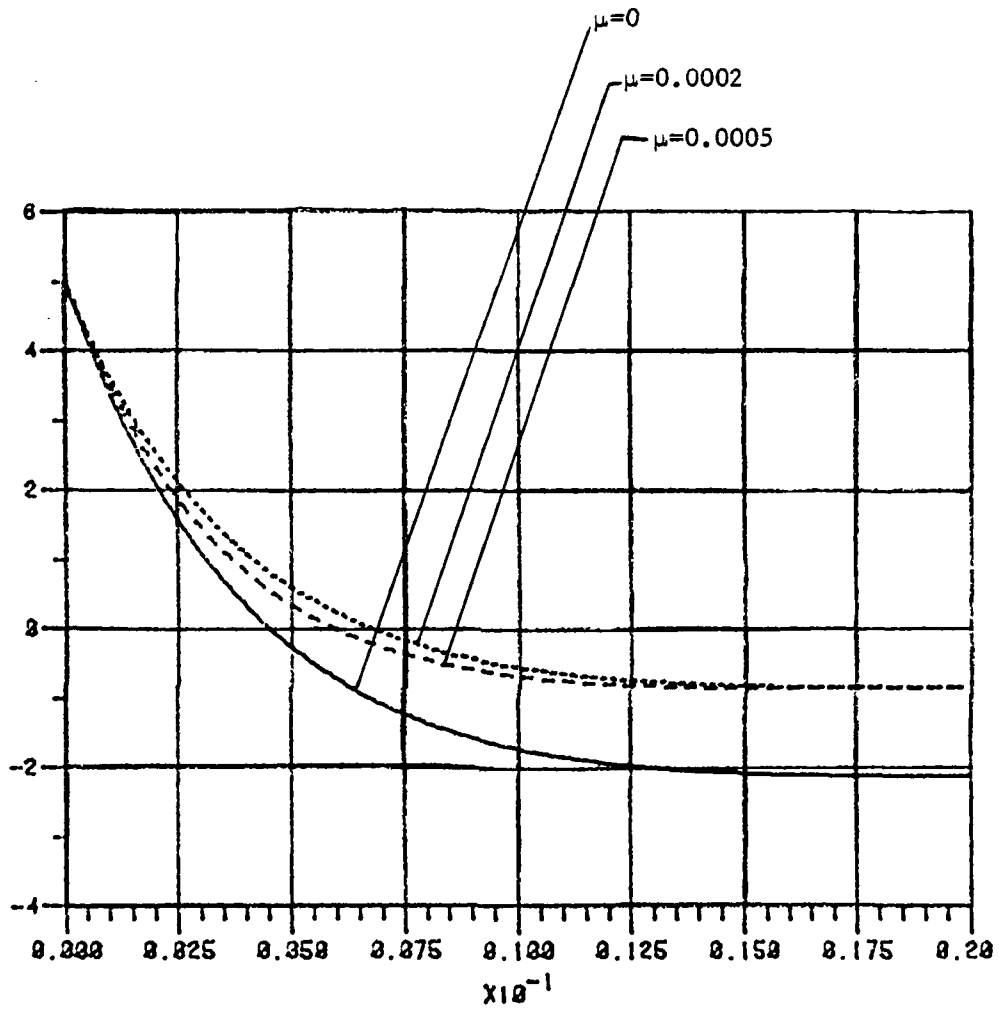


Figure 7.24.  $x_3(t)$  for  $S$  given by equation (7.29)

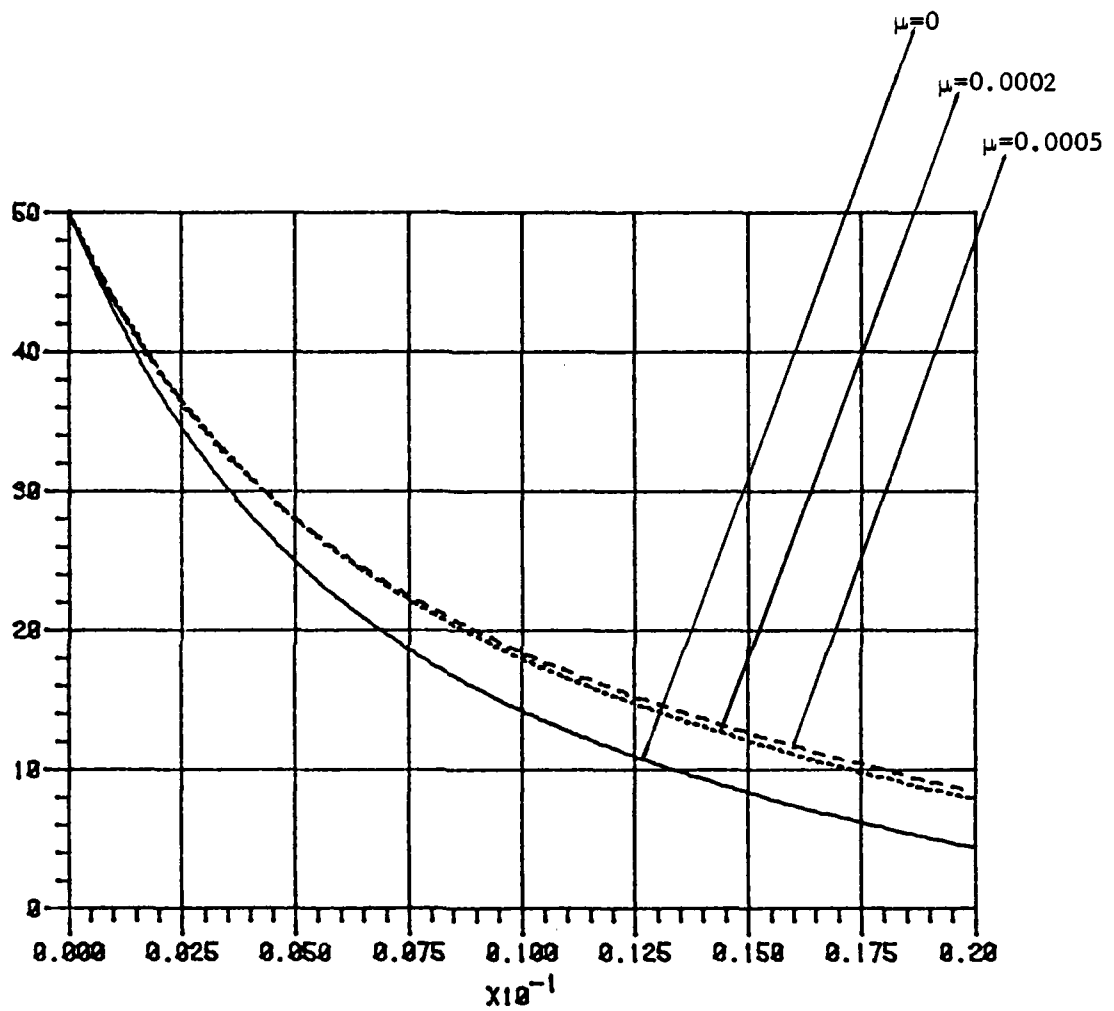


Figure 7.25.  $x_4(t)$  for  $S$  given by equation (7.29)

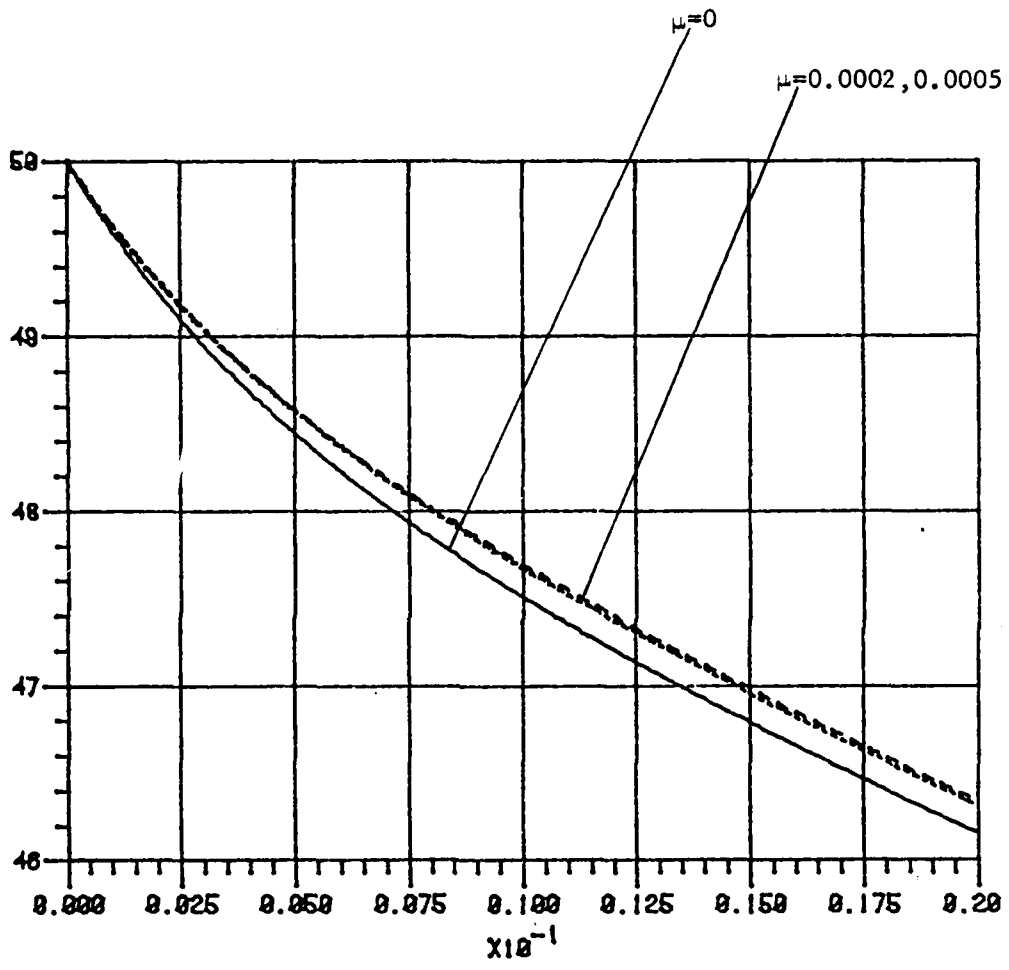


Figure 7.26.  $x_5(t)$  for  $S$  given by equation (7.29)

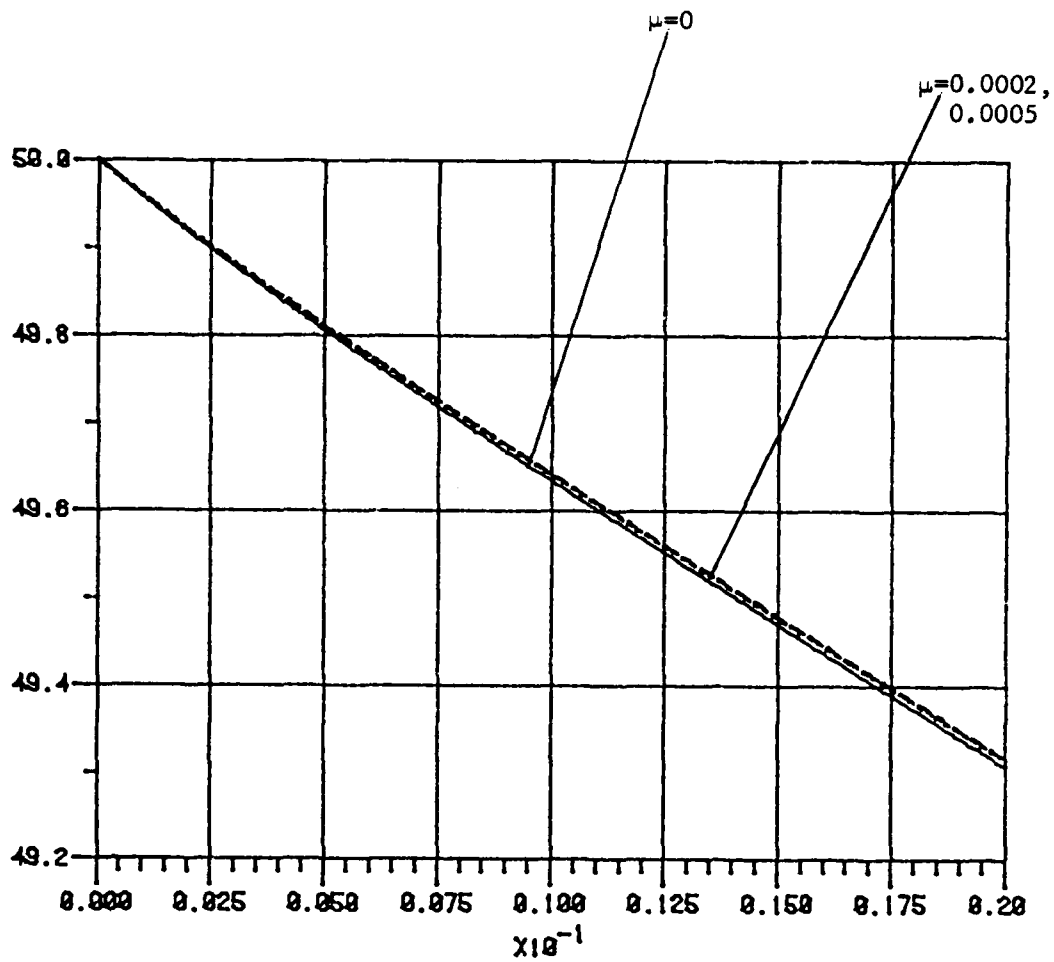


Figure 7.27.  $x_6(t)$  for S given by equation (7.29)



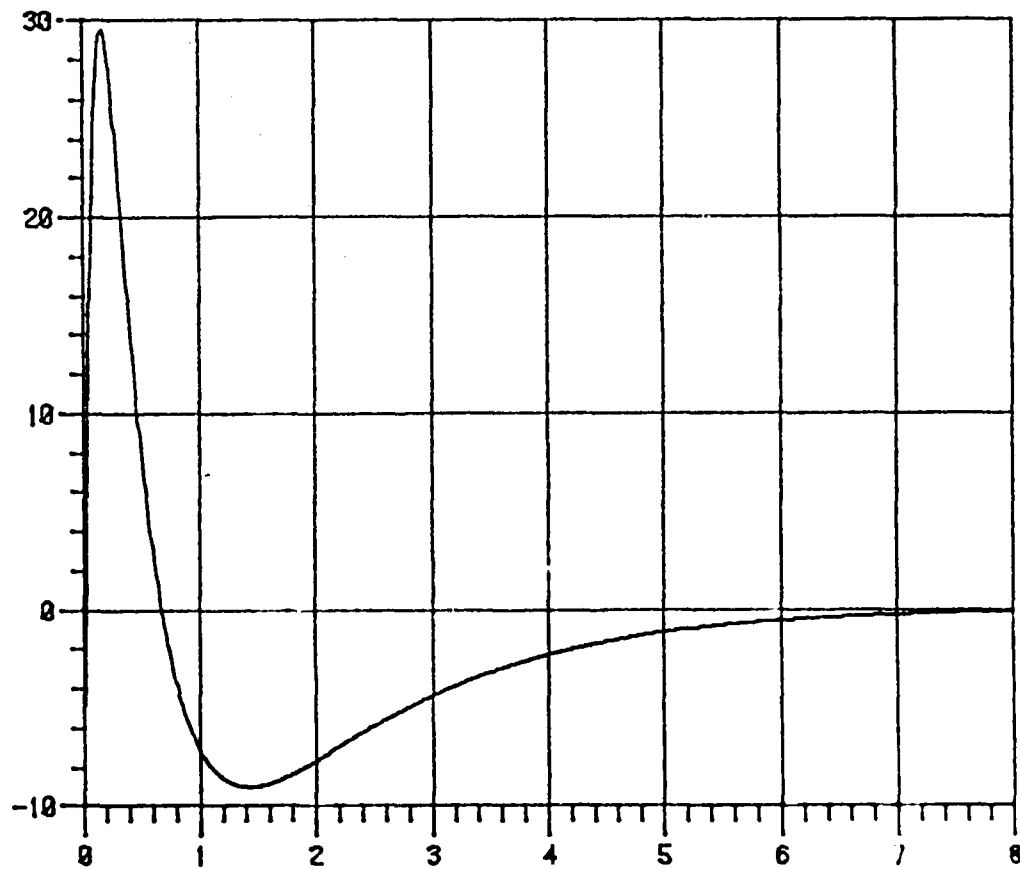


Figure 7.28.  $x_1(t)$  for  $S$  given by equation (7.29) and  $\omega=0$

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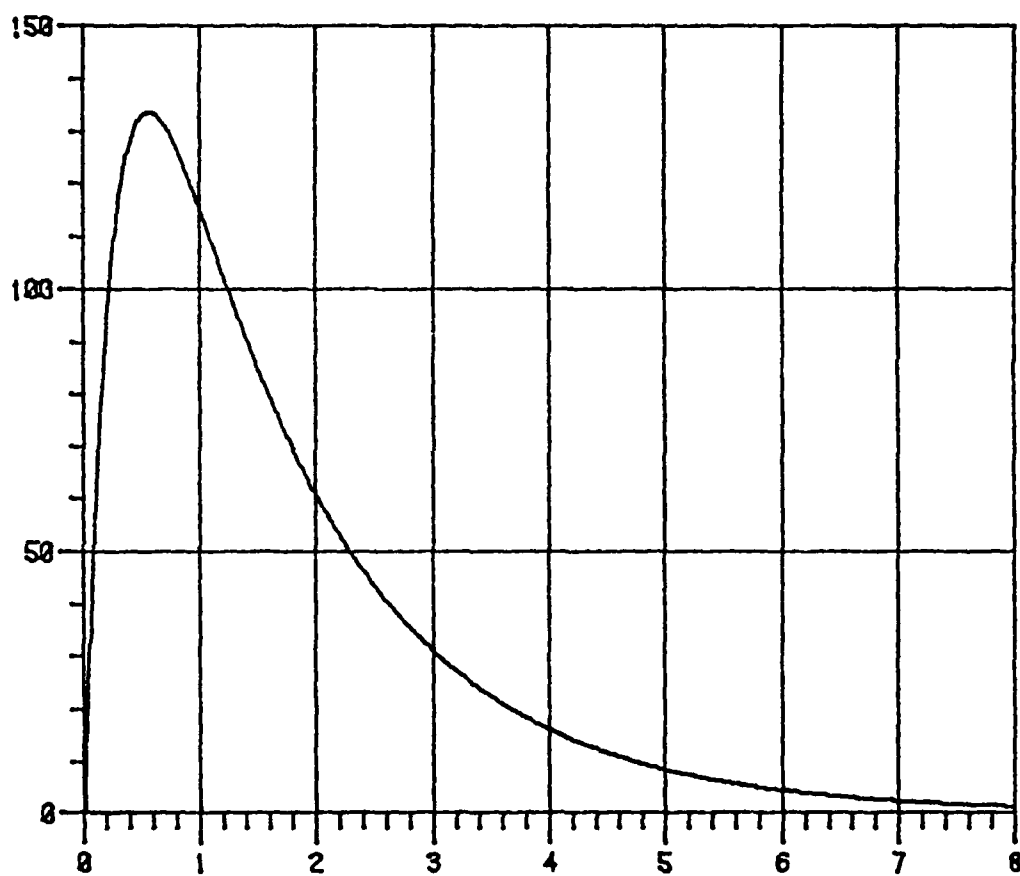


Figure 7.29.  $x_2(t)$  for  $S$  given by equation (7.29) and  $\mu=0$

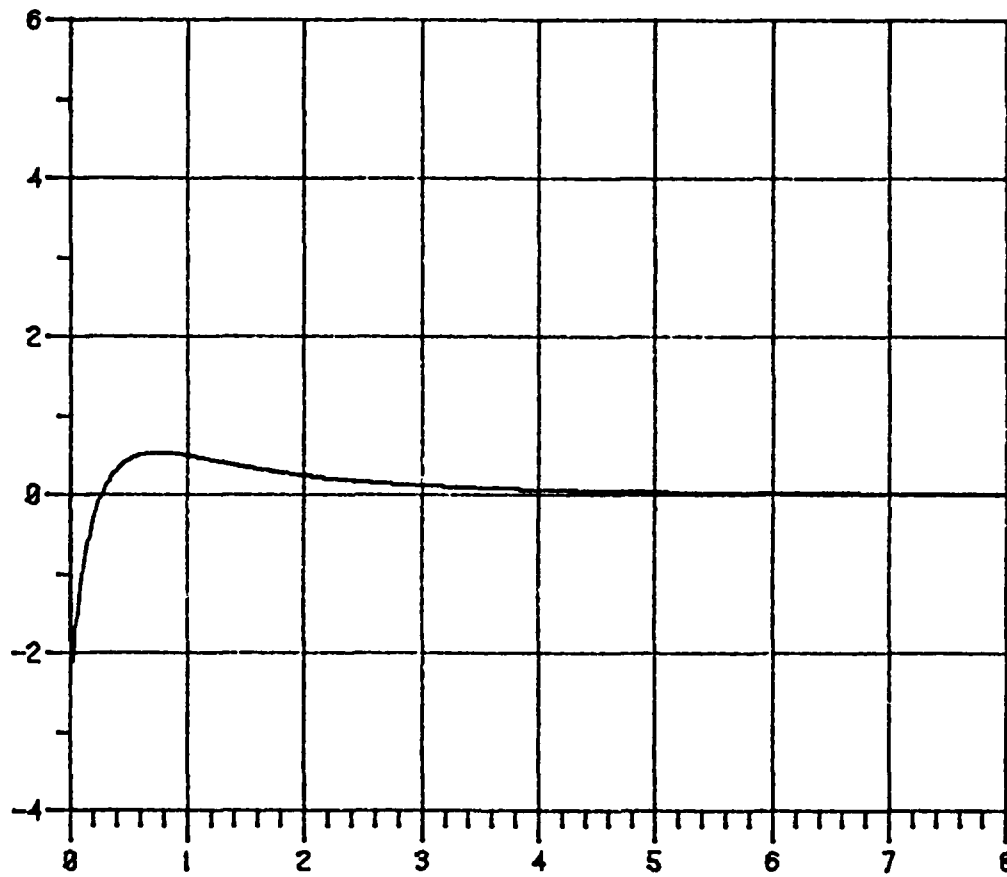


Figure 7.30.  $x_3(t)$  for  $S$  given by equation (7.29) and  $\mu=0$

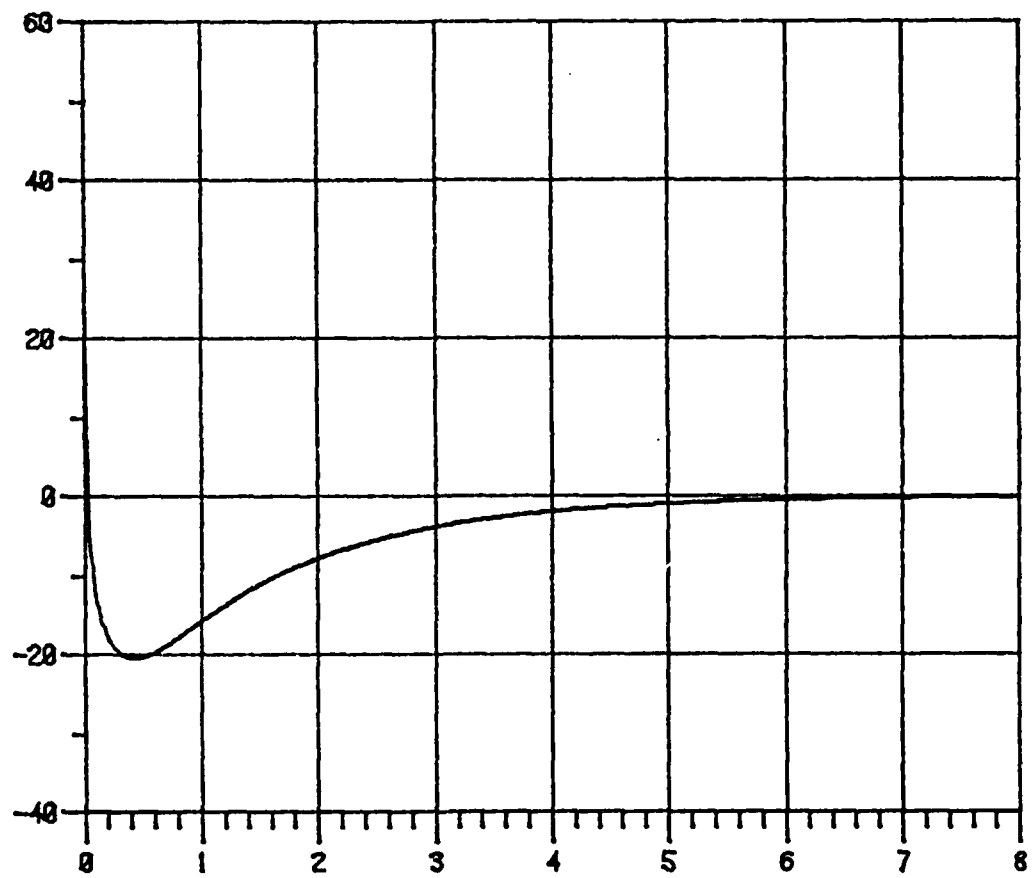


Figure 7.31.  $x_4(t)$  for  $S$  given by equation (7.29) and  $\mu=0$

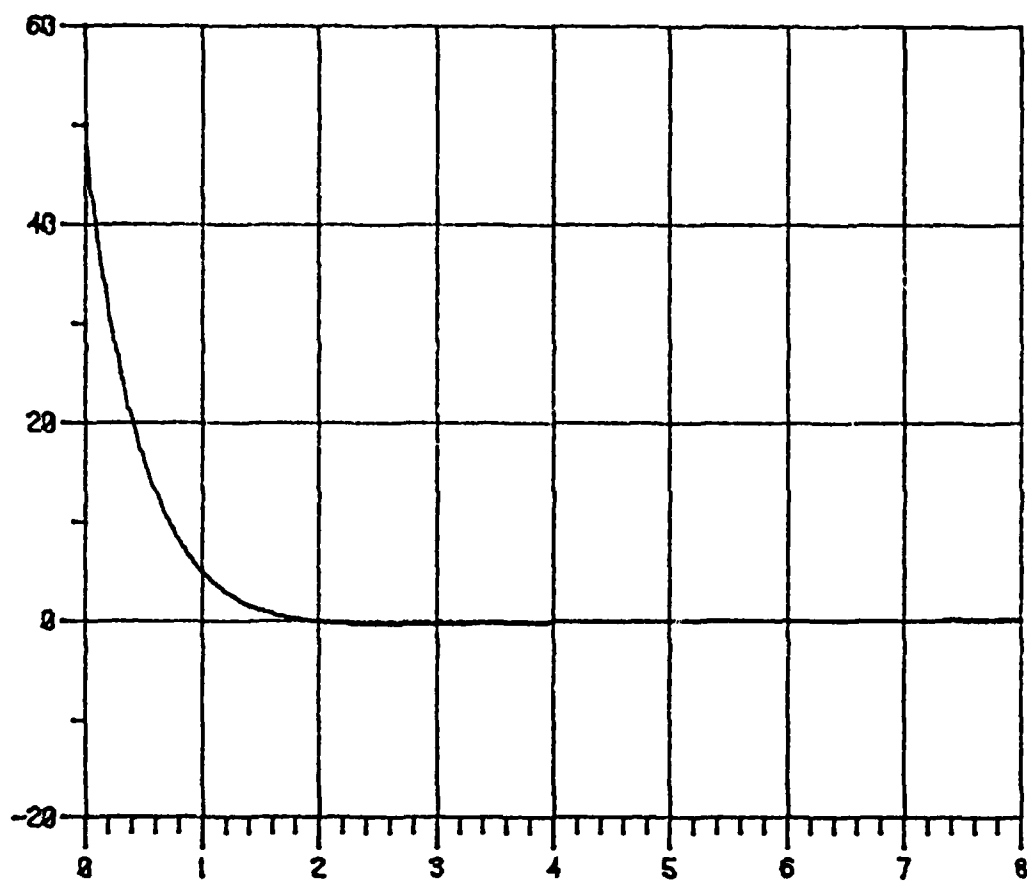


Figure 7.32.  $x_5(t)$  for  $S$  given by equation (7.29) and  $\mu=0$

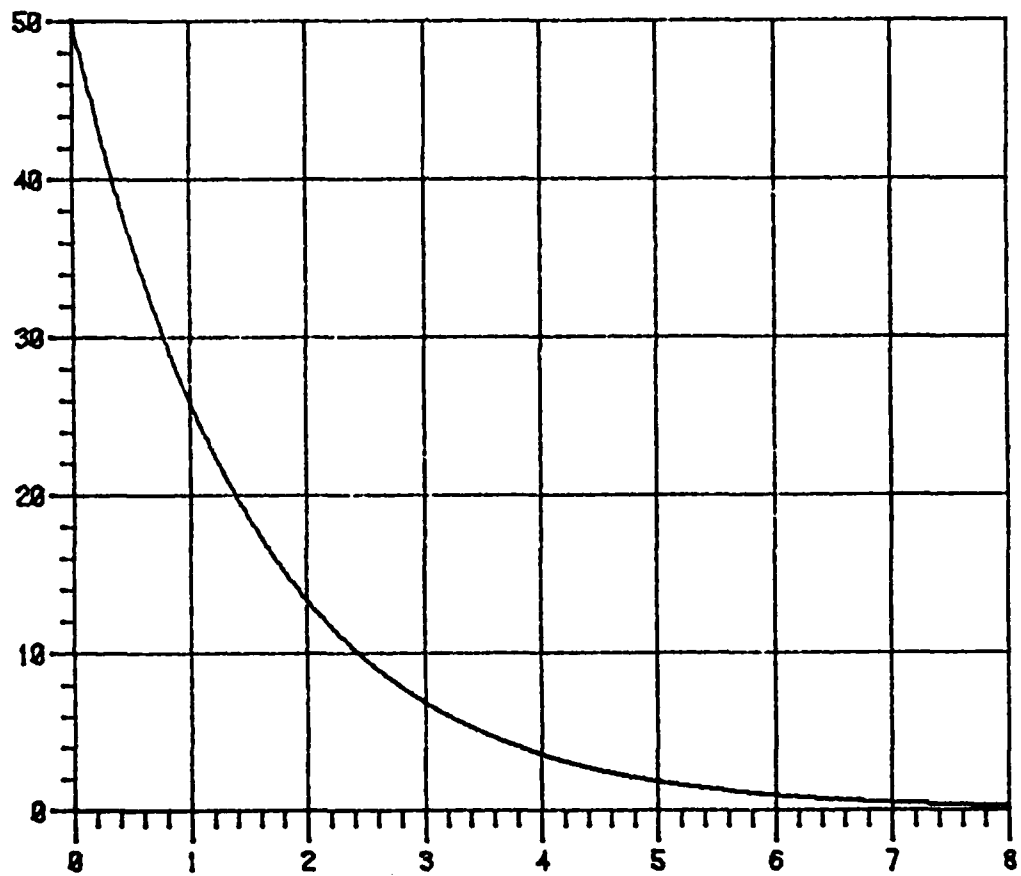


Figure 7.33.  $x_6(t)$  for S given by equation (7.29) and  $\mu=0$

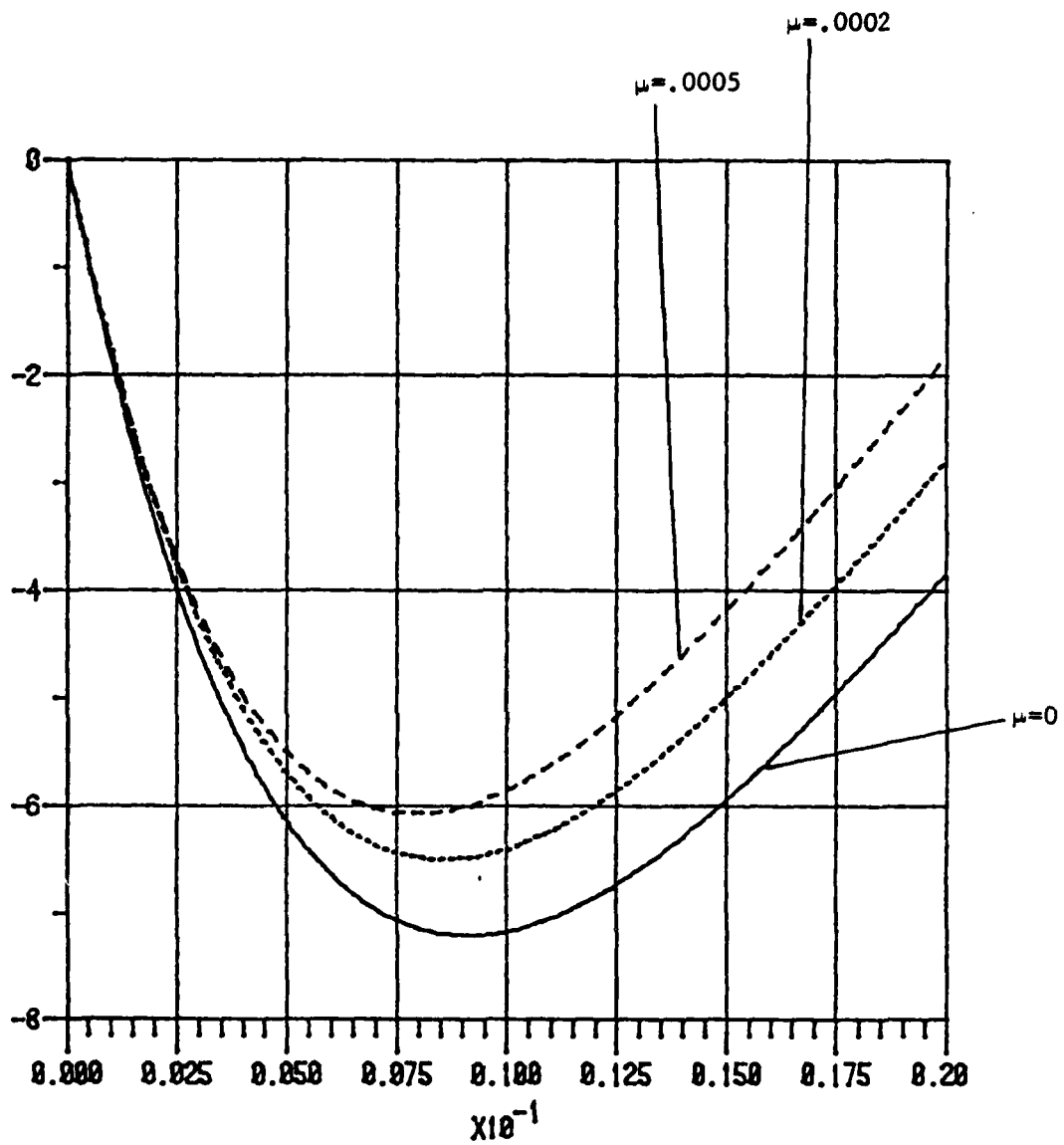


Figure 7.34.  $x_1(t)$  for S given by equation (7.31)



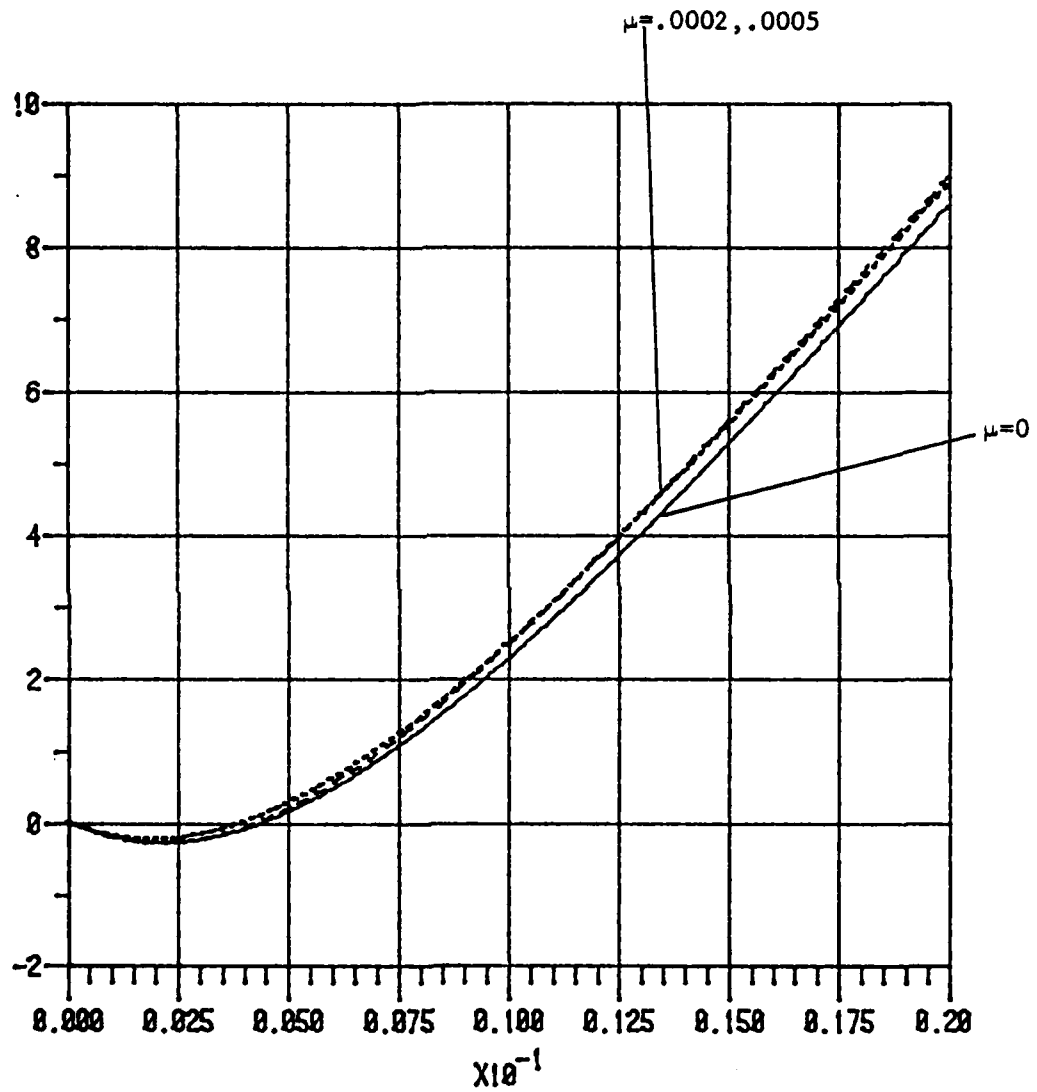


Figure 7.35.  $x_2(t)$  for S given by equation (7.31)

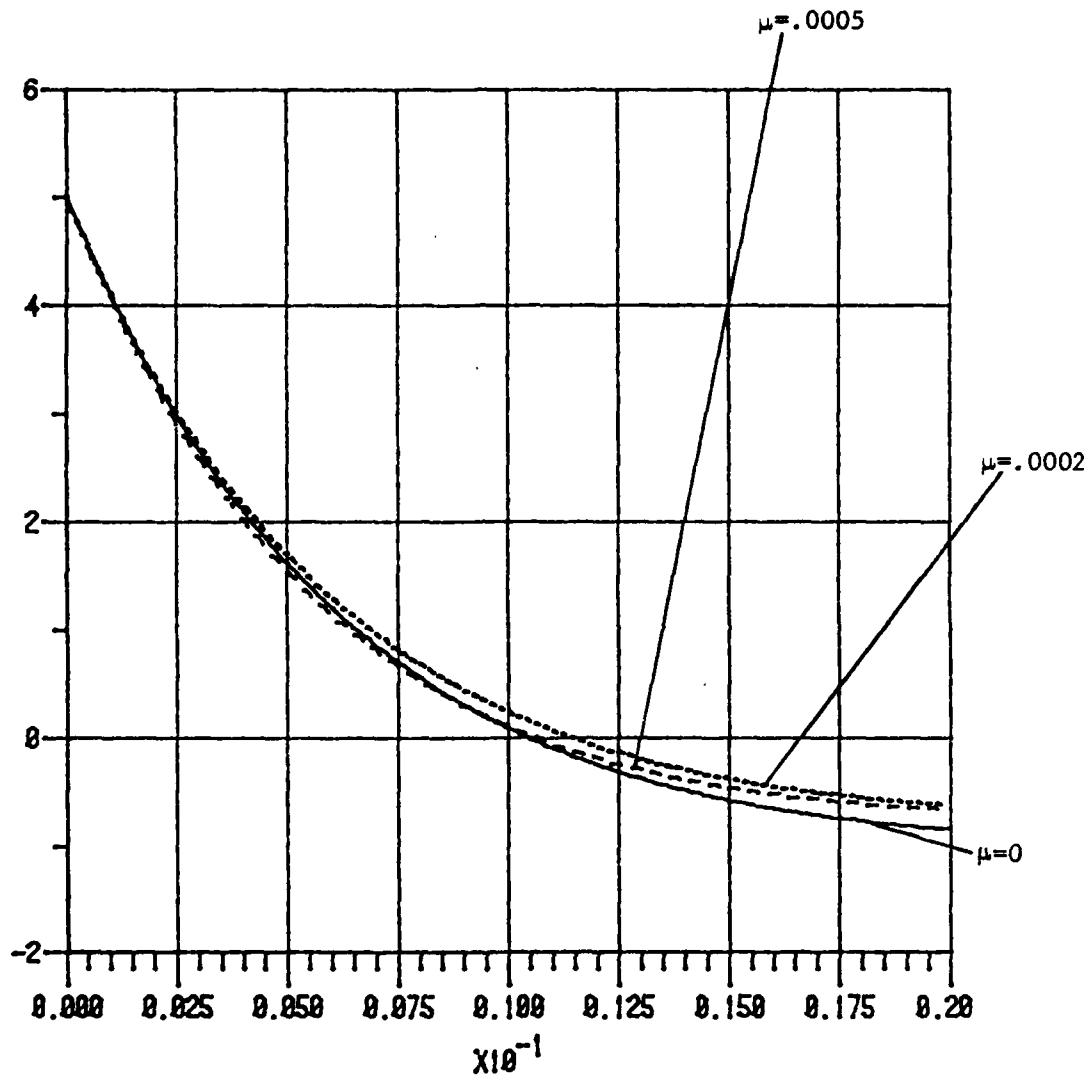


Figure 7.36.  $x_3(t)$  for S given by equation (7.31)

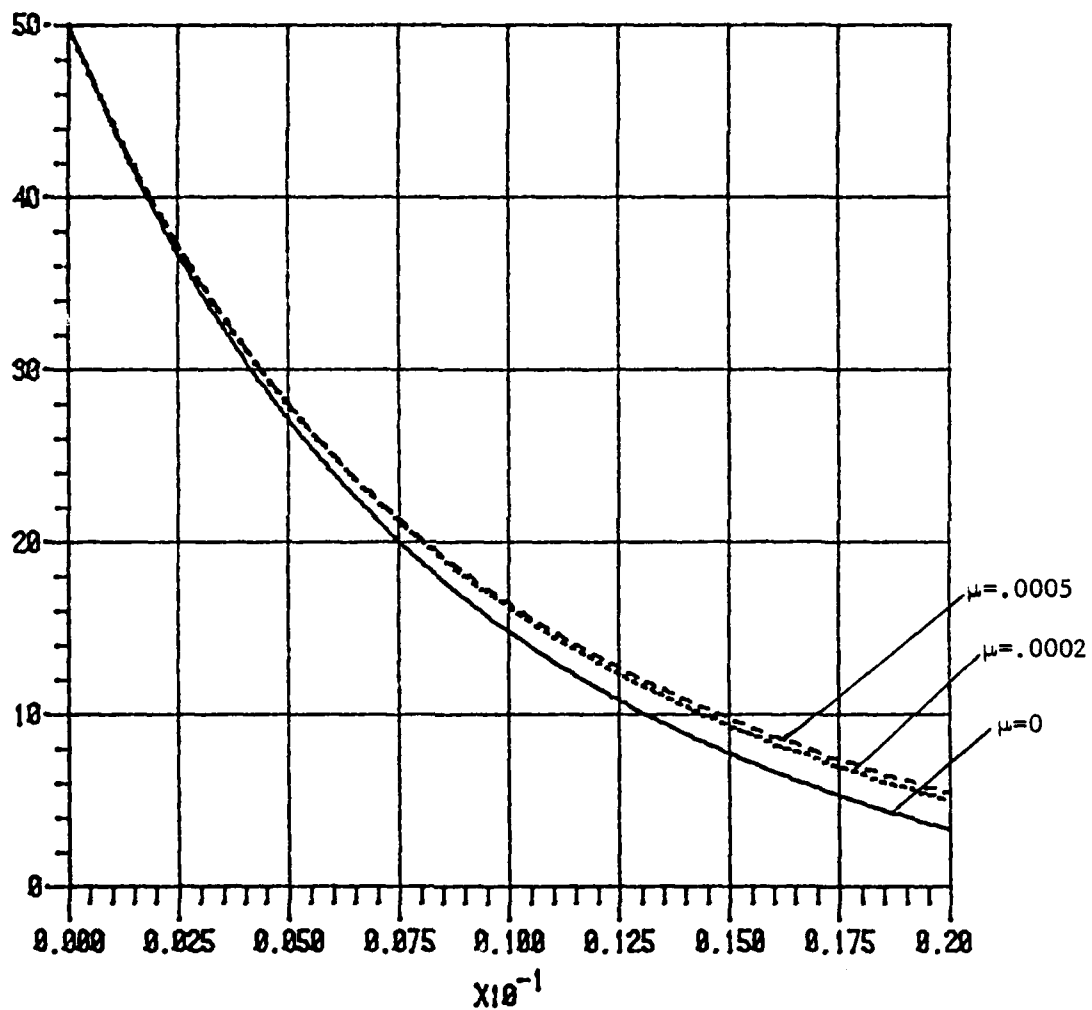


Figure 7.37.  $x_4(t)$  for  $S$  given by equation (7.31)

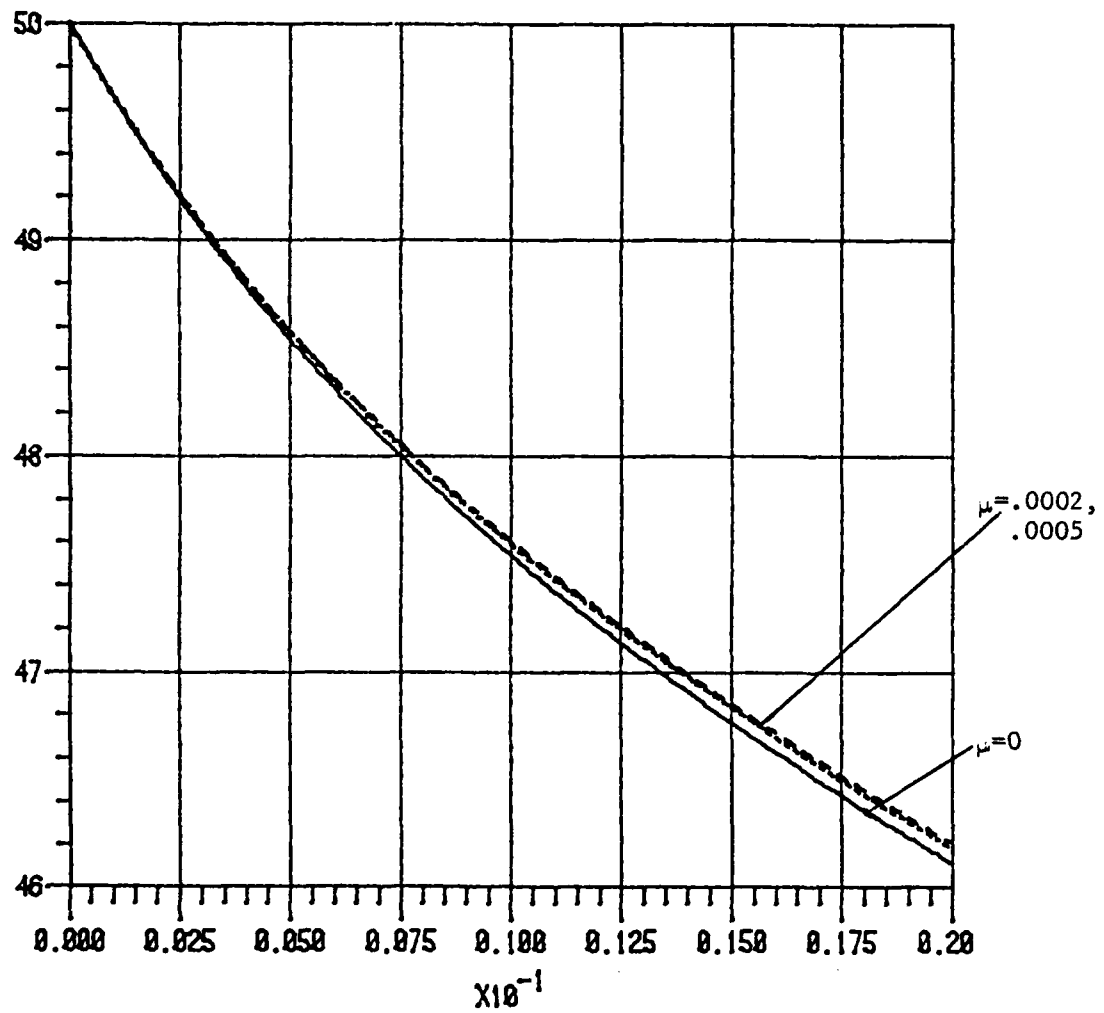


Figure 7.38.  $x_5(t)$  for  $S$  given by equation (7.31)

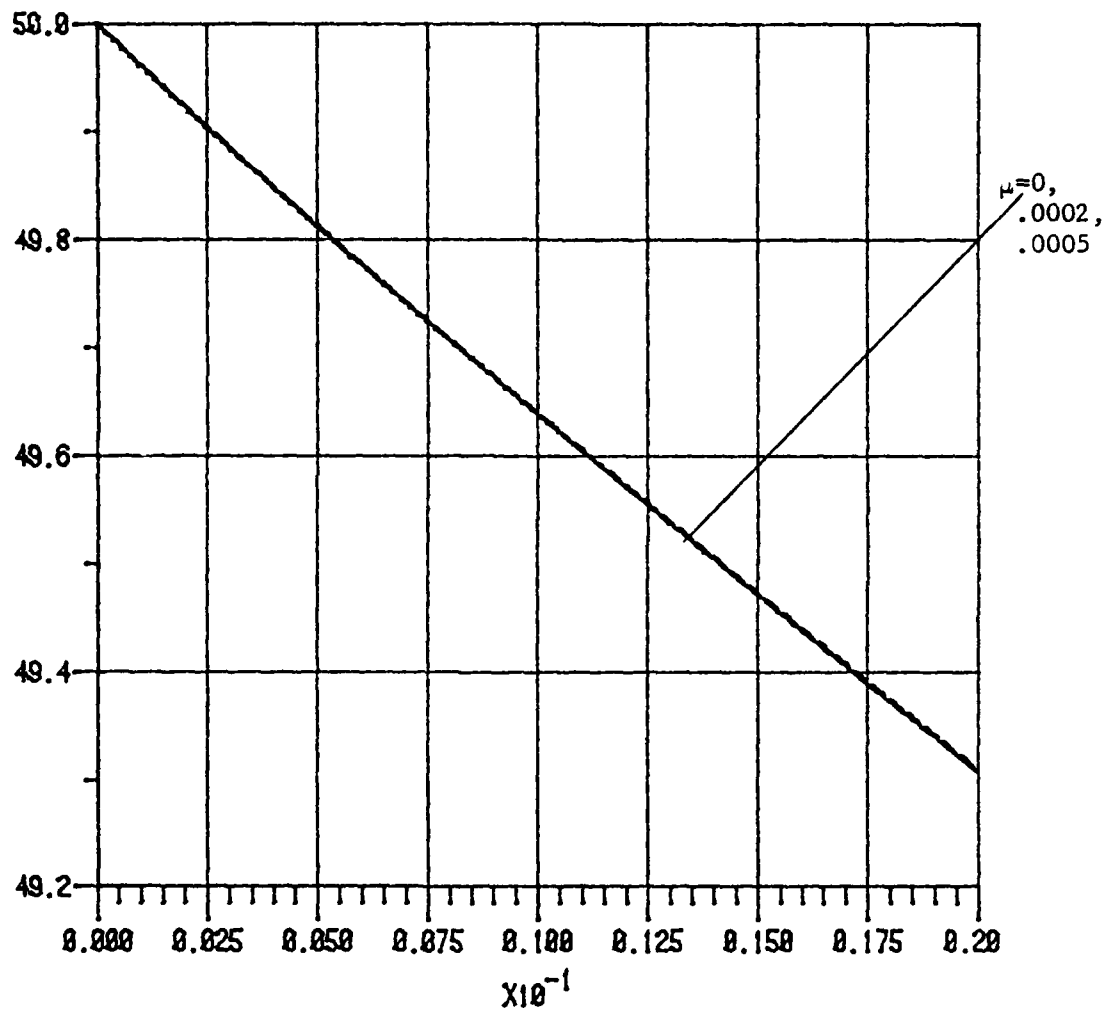


Figure 7.39.  $x_6(t)$  for S given by equation (7.31)

## CHAPTER 8

## SUMMARY AND CONCLUSIONS

The work in this thesis is an addition to the study of trajectory sensitivity of optimal control systems. The parameter with respect to which sensitivity is studied is a small undesirable time delay that might occur in a system designed nominally with zero delays. If the standard linear regulator problem is used, trajectories may show unsatisfactory sensitivities to small time delays. This might be handled by adjusting the design parameters  $Q$  and  $R$ . But that approach may not be convenient because of the lack of a direct theoretical basis for adjusting  $Q$  and  $R$ . In order to provide a direct handle on the sensitivity a design strategy is proposed in which a quadratic performance index which includes a sensitivity measure is minimized. This sensitivity measure is a quadratic term of the sensitivity functions, defined by (2.13), with some weighting matrix  $S$ .  $Q$ ,  $R$  and  $S$  can be adjusted together to get a desirable behavior.  $Q$  penalizes state trajectories,  $R$  penalizes the control and  $S$  penalizes trajectory sensitivity. Necessary conditions of optimality are derived for all possibilities of delay occurrence. Existence of an optimal control is investigated and it is proved that it exists for small values of  $S$ . However, this is only a sufficient condition. In Chapter 7, it has been seen numerically that such a control exists even for relatively large values of  $S$ . Numerical algorithms are presented to solve the resulting necessary conditions of optimality. Convergence of such algorithms is investigated. These algorithms are shown to move in a downhill direction. In other words, the performance index is decreased in each iteration. Analytical proof for convergence is provided for subiterations within these algorithms.

The proposed design strategy is applied to a continuous stirred-tank reactor and a sixth order model of the F100 turbofan engine. It has been seen that the numerical algorithms work efficiently. Several values of  $Q$ ,  $R$ , and  $S$  are tried. It has been seen that the choice of  $S$  with a specific weighting is reflected directly on the sensitivity of the different state trajectories. This direct handle on sensitivity is the main advantage of this proposed design scheme. In both examples suitable values of  $Q$ ,  $R$  and  $S$  are chosen to obtain trajectories with acceptable transient phenomena as well as low sensitivity.

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# VITA

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